Quiz 1

Read each question carefully.

---

1. Where is Croton?

→A. The heel of southern Italy.
B. On the east coast of Sicily.
C. An island off the coast of modern Turkey.
D. None of the above.

---

2. Where is Syracuse?

A. The heel of southern Italy.
→B. On the east coast of Sicily.
C. An island off the coast of modern Turkey.
D. None of the above.

---

3. Where is Alexandria?

A. The heel of southern Italy.
B. On the east coast of Sicily.
C. An island off the coast of modern Turkey.
→D. None of the above.

---

4. What is Thales' main contribution to mathematics?

→A. The introduction of logical proofs in geometry.
B. The first practical applications of geometry.
C. The first accurate prediction of a solar eclipse.
D. All of the above.
E. None of the above.

---

5. What is Plato's main contribution to mathematics?

A. The introduction of logical proofs in geometry.
B. The first practical applications of geometry.
C. The first accurate prediction of a solar eclipse.
D. All of the above.
→E. None of the above.

Quiz 2

Read each question carefully.

---

1. According to Kolmogorov, where do mathematical discoveries arise?

A. Beyond the impossible.
B. Amongst the trivial.
→*C*. Between the trivial and the impossible.
D. None of the above.

---

2. How many subjects of the classical Greek curriculum are mathematical?

A. One.
→*B*. Two.
C. Four. *(Based on class discussions I accept this answer.)*
D. None of the above.

---

3. What is the Weber-Fechner law?

→*A*. Our ears perceive multiplication as addition.
B. Our ears perceive addition as multiplication.
C. Our ears perceive multiplication and addition as the same.
D. None of the above.

---

4. In music, what is a fifth?

A. Notes whose frequencies are $2:1$.
→*B*. Notes whose frequencies are $3:2$.
C. Notes whose frequencies are $5:1$.
D. None of the above.

---

5. What is a pythagorean triple?

A. Three notes in a succession of fifths.
B. Three notes in a major chord.
C. The gap between seven fifths and five octaves.
→*D*. None of the above.

Quiz 3

Read each question carefully.

---

1. How did the Egyptians represent fractional amounts?

A. Decimals.
B. Common fractions.
$\rightarrow C$. Unit fractions.
D. All of the above.
E. None of the above.

---

2. What was the basis for the Egyptian algorithms for multiplication and addition?

A. Decimal arithmetic.
B. Sexagesimal arithmetic.
$\rightarrow C$. Binary arithmetic.
D. None of the above.

---

3. What does the euclidean algorithm yield?

A. A common measure for two magnitudes.
B. The greatest common divisor of two whole numbers.
$\rightarrow C$. Both.
D. Neither.

---

4. What is significant about the continued fraction for $1 + \sqrt{2}$?

A. It is infinite.
B. It is periodic.
$\rightarrow C$. Both.
D. Neither.

---

5. What is significant about the relationship between the diagonal of a square and its side?

A. They are incommensurable.
B. Their ratio is irrational.
$\rightarrow C$. Both.
D. Neither.

Quiz 4

Read each question carefully.

---

1. What is the continued fraction for 48/11?
→*A*. $4 + /2 + /1 + /3$
B. $4 + /2 + /1 + /3 + /4 + /2 + /1 + /3 + \cdots$
C. $4 + /2 + /1 + /3 + /3 + /3 + /3 + /3 + \cdots$
D. None of the above.

---

2. What is the continued fraction for $\sqrt{2}$?
A. $1 + /2 + /1 + /2 + \cdots$
B. $1 + /1 + /1 + /1 + \cdots$
→*C*. $1 + /2 + /2 + /2 + \cdots$
D. None of the above.

---

3. What is the binary representation of 9?
A. 4 1
B. 1 1
C. 1 0 1
→*D*. None of the above.

---

4. What is the sexagesimal representation of 68?
→*A*. 1 8
B. 1 2
C. 1 1 2
D. None of the above.

---

5. How would the ancient Egyptians have expressed the double of one fifth?
A. $\frac{2}{5}$
B. $\frac{1}{5} + \frac{1}{5}$
C. $\frac{1}{2} + \frac{1}{5}$
→*D*. None of the above.

Quiz 5

Read each question carefully.

---

1. What good is the discriminant of a quadratic equation?

A. It is part of the formula for the solution for the equation.
B. It determines whether there are any real solutions.
C. It determines whether any of the solutions are imaginary.
→*D*. All of the above.
E. None of the above.

---

2. What is involved in applying Cardano's formula for cubic equations?

A. Introducing two auxiliary variables.
B. Reducing a system of two equations to a sextic.
C. Applying the quadratic formula to a sextic.
→*D*. All of the above.
E. None of the above.

---

3. What was Bombelli's contribution to the solution of cubics?

→*A*. The reconciliation of real solutions via imaginary numbers.
B. The interpretation of imaginary solutions using triangles.
C. The application of Cardano's formula to quartics.
D. All of the above.
E. None of the above.

---

4. Where were imaginary numbers before 1572?

A. In the pythagorean arithmetic of continued fractions.
B. In the ancient Iraqi tablet called Plimpton 322.
C. In the ancient Egyptian document called the Rhind papyrus.
D. All of the above.
→*E*. None of the above.

---

5. How do we multiply complex numbers?

A. Add the lengths and multiply the angles.
→*B*. Multiply the lengths and add the angles.
C. Multiply the lengths and multiply the angles.
D. None of the above.

Quiz 6

Read each question carefully.

---

1. How would the ancient Iraqis express the solution to the problem of finding a pair of numbers given their sum and difference?

A. The sum plus or minus half the difference.
→*B*. Half the sum plus or minus half the difference.
C. Half the sum plus or minus the difference.
D. None of the above.

---

2. How would the ancient Iraqis solve the problem of finding a pair of numbers given their sum and product?

→*A*. By reducing it to the type of problem above.
B. By application of what later became known as "Pythagoras' theorem".
C. By transposing all terms to one side and then factoring the quadratic.
D. All of the above.
E. None of the above.

---

3. How can we frame the problem of finding pythagorean triples geometrically?

A. Using elliptic curves.
→*B*. Using the unit circle.
C. Using chakravala.
D. All of the above.
E. None of the above.

---

4. How did Brahmagupta and Bhaskaracharya solve what we now call "Pell's equation"?

A. Using elliptic curves.
B. Using the unit circle.
→*C*. Using chakravala.
D. All of the above.
E. None of the above.

---

5. How did Pacioli solve cubic and quartic equations?

A. Using elliptic curves.
B. Using the unit circle.
C. Using chakravala.
D. All of the above.
→*E*. None of the above.

Quiz 7

Read each question carefully.

---

1. At the end of the Middle Ages where did Europe first learn about algebraic methods?
→*A*. Arabic texts in reconquered Spain.
B. Texts brought back from India by Portuguese Jesuits.
C. Texts brought back from China by Marco Polo.
D. All of the above.
E. None of the above.

---

2. What have been the most important consequences of Cardano's formulas for solving cubics and quartics?
A. General methods for solving all algebraic equations.
B. Efficient algorithms for approximating the solutions of algebraic equations.
→*C*. An understanding of the abstract structure of polynomials and polynomial equations.
D. All of the above.
E. None of the above.

---

3. How soon after Cardano's formulas were analogous formulas found for quintic equations?
A. Within 100 years.
B. About 200 years afterwards.
C. Only very recently.
→*D*. None of the above.

---

4. How did the work of Galois and his contemporaries change algebra?
A. Algebra took on its current symbolic form.
→*B*. Algebra's focus turned from algorithm to abstraction.
C. Algebraists adopted the methodologies developed in the Islamic world.
D. All of the above.
E. None of the above.

---

5. Whose work showed the limitations of Hilbert's program for the algebraization of all mathematics?
A. Emil Post.
B. Alonzo Church.
C. Kurt Goedel.
→*D*. All of the above.
E. None of the above.

Quiz 8

Read each question carefully.

---

1. Who first came to grips with the *casus irreducibilis*?
  A. Nicole Oresme.
  B. Girolamo Cardano.
$\rightarrow C$. Rafael Bombelli.
  D. None of the above.

---

2. Who developed the method for computing n-th roots using complex numbers?
$\rightarrow A$. Abraham de Moivre.
  B. Sir Isaac Newton.
  C. Leonhard Euler.
  D. None of the above.

---

3. What is $|3 + 2i|$?
  A. $3 + 2$.
  B. $3 - 2i$.
  C. $3^2 - 2^2$.
$\rightarrow D$. None of the above.

---

4. What is Euler's identity?
  A. $e^{\theta i} = e^{\theta} + i$.
  B. $e^{\theta i} = e^{\theta} + e^{i}$.
  C. $e^{\theta i} = e^{\theta} \cdot e^{i}$.
$\rightarrow D$. None of the above.

---

5. Where lies the shortest path between two truths in the real domain?
  A. In the fine layer between the trivial and the impossible.
$\rightarrow B$. In the complex domain.
  C. In the world of abstraction.
  D. None of the above.

Quiz 9

Read each question carefully.

---

1. Which homogeneous coordinates determine the same projective point as $(2, 3, 1)$?
A. $(4, 6, 2)$
B. $(-2, -3, -1)$
C. $(1, 3/2, 1/2)$
$\rightarrow D$. All of the above.
E. None of the above.

---

2. The *costruzione legittima* works because which things remain the same in any view of the plane?
A. Straight lines remain straight.
B. Intersections remain intersections.
C. Parallel lines remain parallel or meet at the horizon.
$\rightarrow D$. All of the above.
E. None of the above.

---

3. Which of the following are models for the points of the projective plane?
A. Lines through the origin in 3-space.
B. Pairs of antipodal points on the sphere.
C. Points of the euclidean plane together with the points on a line at infinity.
$\rightarrow D$. All of the above.
E. None of the above.

---

4. Which of the following are consequences of Euclid's parallel axiom?
A. Rectangles exist.
B. The pythagorean theorem.
C. The angles of any triangle sum to two right angles.
$\rightarrow D$. All of the above.
E. None of the above.

---

5. What is the first great advance in geometry after the time of Euclid?
A. The introduction of coordinates.
B. Apollonios' work on conic sections.
C. Diophantos' work on elliptic curves.
D. The development of projective geometry.
E. None of the above.

Quiz 10

Read each question carefully.

---

1. What are the homogeneous coordinates of the points at infinity on the hyperbola $x^2 = y^2 + 1$?

A. $(1, 1, 0)$
B. $(1, -1, 0)$
→C. Both.
D. Neither.

---

2. What is a model of neutral geometry?

A. Euclidean plane.
B. Projective plane.
→C. Both.
D. Neither.

---

3. How do you construct the projective plane from the sphere?

→A. Glue together antipodal points.
B. Delete one hemisphere and flatten the other.
C. Add a line at infinity.
D. None of the above.

---

4. What was the first theorem on projective geometry?

A. Pythagoras' theorem.
→B. Pappus' theorem.
C. Desargues's theorem.
D. None of the above.

---

5. Which laws of algebra are consequences of Desargues's theorem?

→A. All but the commutative law of multiplication.
B. All but the associative law of multiplication.
C. Only the commutative and associative laws of multiplication.
D. None of the above.

Quiz 11

Read each question carefully.

---

1. What does Diophantos' secant-and-tangent method yield when applied to the unit circle?

A. A proof of the pythagorean theorem.
B. A resolution of the pythagorean comma.
C. A refutation of the "pythagorean dream".
D. All of the above.
→*E*. None of the above.

---

2. What does Diophantos' secant-and-tangent method yield when applied to a cubic curve?

A. A rational parameterization of the curve.
B. All rational points on the curve.
→*C*. A law of composition for rational points on the curve.
D. All of the above.
E. None of the above.

---

3. If C is an elliptic curve and if P and Q are points on C then what can we say about $P * Q$?

→*A*. P, Q, and $P * Q$ are collinear.
B. P, Q, and $P * Q$ are not collinear.
C. $P * Q$ is equidistant from P and Q.
D. None of the above.

---

4. If C is an elliptic curve and if P is a point on C then what do we use to determine $P * P$?

A. The line at infinity.
→*B*. The tangent line to C at P.
C. The circle centered at P and tangent to the line at infinity.
D. None of the above.

---

5. How did Diophantos determine tangent lines?

A. Using differential calculus.
B. Using integral calculus.
C. Both.
→*D*. Neither.

Quiz 12

Read each question carefully.

---

1. Suppose we apply the ancient Iraqi method for computing $\sqrt{7}$, and start with the approximation 2. What does the next iteration produce?

A. $\frac{1}{2}(2+7)$

→*B*. $\frac{1}{2}(2+7/2)$

C. $\frac{1}{2}(2/7+7/2)$

D. None of the above.

---

2. How does the accuracy improve as the ancient Iraqi method for computing square roots proceeds?

A. We obtain roughly 1 extra decimal place of accuracy at each iteration.
B. We obtain roughly 10 extra decimal places of accuracy at each iteration.
C. We obtain roughly 100 extra decimal places of accuracy at each iteration.
→*D*. None of the above.

---

3. Who first discovered Pascal's triangle?

A. Pascal.
B. Horner.
C. Eudoxus of Cnidos.
→*D*. None of the above.

---

4. Who first discovered Horner's method?

A. Pascal.
B. Horner.
C. Eudoxus of Cnidos.
→*D*. None of the above.

---

5. Who first discovered the method of exhaustion?

A. Pascal.
B. Horner.
→*C*. Eudoxus of Cnidos.
D. None of the above.

Quiz 13

Read each question carefully.

---

1. What distinguishes the concept of "infinitesimal" from that of "arbitrarily small"?

→*A*. An infinitesimal is a single object that is smaller than any measurable amount but does not vanish.
B. An infinitesimal is a series of objects that become smaller than any pre-assigned amount.
C. An infinitesimal is a single object that is larger than any measurable amount but is not infinite.
D. An infinitesimal is a series of objects that become larger than any pre-assigned amount.
E. None of the above.

---

2. How did the Greeks define equality of rectangles?

A. Rectangles are equal if and only if they are congruent.
→*B*. Rectangles are equal if and only if one can be cut by finitely many straight lines and re-assembled to form the other.
C. Rectangles are equal if and only if one can be cut into infinitely many infinitesimal pieces and re-assembled to form the other.
D. None of the above.

---

3. If $r \neq 1$ then what is $a + ar + ar^2 + \cdots + ar^n$?

A. ar^{n+1}
B. $\dfrac{ar^{n+1}}{n+1}$
C. $(n+1)ar^{n+1}$
→*D*. None of the above.

---

4. If $|r| < 1$ then what is $a + ar + ar^2 + \cdots$?

→*A*. $\dfrac{a}{1-r}$
B. $a(1-r)$
C. $ar(1-r)$
D. None of the above.

---

5. What is $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$?

A. $\dfrac{1}{\pi}$
→*B*. $\dfrac{\pi}{4}$
C. $\dfrac{\pi}{\sqrt{2}}$
D. None of the above.

Quiz 14

Read each question carefully.

---

1. If $|r| < 1$ then what is $a - ar + ar^2 - ar^3 + \cdots$?

$A.$ $\frac{a}{1-r}$

$\rightarrow B.$ $\frac{a}{1+r}$

$C.$ $\frac{a}{1-r^2}$

$D.$ None of the above.

---

2. If $|r| < 1$ then what is $ar + ar^2 + ar^3 + \cdots$?

$A.$ $\frac{a}{1-r}$

$\rightarrow B.$ $\frac{ar}{1-r}$

$C.$ $\frac{a}{r-r^2}$

$D.$ None of the above.

---

3. If $|r| < 1$ then what is $a + ar^2 + ar^4 + ar^6 + \cdots$?

$\rightarrow A.$ $\frac{a}{1-r^2}$

$B.$ $\frac{ar}{1-r}$

$C.$ $\frac{a}{r-r^2}$

$D.$ None of the above.

---

4. If $y = x^2$ then what is dy?

$\rightarrow A.$ $2x\,dx + (dx)^2$

$B.$ $2x\,dx$ Note: In the interpretation of dx in chapter 4 dy is only *adequal* to $2x\,dx$, not equal to it. Of course the point of the chapter is the "yearning" for answer B. For this one needs to argue that $(dx)^2 = 0$.

$C.$ $2x$

$D.$ None of the above.

---

5. What is the difference between $2x$ and $2x + dx$?

$A.$ A small but measurable amount.

$\rightarrow B.$ An infinitesimal amount.

$C.$ Nothing at all.

Quiz 15

Read each question carefully.

---

1. What does Stillwell claim about the conception of our world in ancient times?

A. They believed that the earth is flat but space is round.
B. They believed that the earth is round but space is flat.
→*C*. They believed that the earth is round and so is space.
D. None of the above.

---

2. What did Lucretius believe about space?

A. It is round but finite.
B. It is flat but finite.
→*C*. It is flat and infinite.
D. None of the above.

---

3. What is the geometry of space implicit in Dante's *Paradiso*?

→*A*. Spherical.
B. Euclidean.
C. Projective.
D. None of the above.

---

4. What does stereographic projection achieve?

A. A projection of the 3-sphere onto the 2-sphere.
→*B*. A projection of the 2-sphere onto the flat plane.
C. A projection of the 3-sphere onto the flat plane.
D. None of the above.

---

5. What is true about the curvature of the cylinder?

→*A*. It is intrinsically flat but extrinsically curved.
B. It is extrinsically flat but intrinsically curved.
C. It is both intrinsically and extrinsically curved.
D. None of the above.

Quiz 16

Read each question carefully.

---

1. Which property of geodesics on a sphere indicates curvature?
- A. They are finite closed curves.
- B. Certain pairs of points are connected by more than one geodesic.
- $\rightarrow C$. There are no parallels.
- D. All of the above.
- E. None of the above.

---

2. How does the sum of the angles in a spherical triangle differ from π?
- A. It exceeds π by an amount proportional to the perimeter.
- $\rightarrow B$. It exceeds π by an amount proportional to the area.
- C. It is less than π by an amount proportional to the area.
- D. None of the above.

---

3. Which of the following is *not* a property of noneuclidean geometry?
- A. Lines are infinite.
- B. There is a unique line through any two points.
- $\rightarrow C$. There is a unique parallel to any line through a point outside it.
- D. All of the above.
- E. None of the above.

---

4. Who discovered noneuclidean geometry?
- A. Gauss.
- B. Bolyai.
- C. Lobachevsky.
- $\rightarrow D$. All of the above.
- E. None of the above.

---

5. What did Saccheri contribute to geometry?
- A. The discovery of asymptotic lines in noneuclidean geometry.
- B. The proof that asymptotic lines meet the line at infinity in right angles.
- C. The proof that Euclid's first four postulates suffice to prove that parallels exist.
- $\rightarrow D$. All of the above.
- E. None of the above.

Quiz 17

Read each question carefully.

---

1. What is true about the tractrix?

A. It is the curve traced by a weight dragged behind a tractor.
B. It generates a surface of revolution of constant negative curvature.
C. It generates a surface of revolution whose triangles obey the laws of noneuclidean geometry.
→*D*. All of the above.
E. None of the above.

---

2. What did Beltrami prove about how surfaces of constant curvature can be mapped onto the plane?

A. It can be done in a way that preserves angles.
→*B*. It can be done in a way that maps geodesics to straight lines.
C. It can be done in a way that faithfully represents distances between points.
D. All of the above.
E. None of the above.

---

3. What is a conformal map?

→*A*. One that preserves angles.
B. One that maps geodesics to straight lines.
C. One that faithfully represents distances between points.
D. All of the above.
E. None of the above.

---

4. What does Gunn's picture clearly reveal about hyperbolic space?

A. A curve equidistant from a line is itself a line.
B. A line in hyperbolic space really is straight.
C. The sky would look better in black.
D. All of the above.
→*E*. None of the above.

---

5. What is the geometry of real physical space?

A. Euclidean.
B. Spherical.
C. Hyperbolic.
→*D*. Variable.

Quiz 18

Read each question carefully.

---

1. Suppose we apply the ancient Iraqi method for computing $\sqrt{3}$, and start with the approximation 2. What does the next iteration produce? (Answer in sexagesimal!)

   A. $1;15$
   B. $1;30$
→*C*. $1;45$
   D. None of the above.

---

2. Suppose we continue with the calculation above for 5 more iterations. Roughly how many accurate sexagesimal digits would we obtain?

   A. Around 5 or 6.
→*B*. A few dozen.
   C. A few hundred.
   D. None of the above.

---

3. If we apply "synthetic division" to a polynomial $f(x)$ and a value a then what do we obtain?

   A. The quotient of $f(x) \div (x-a)$.
   B. The remainder of $f(x) \div (x-a)$.
   C. The value $f(a)$.
→*D*. All of the above.
   E. None of the above.

---

4. If we use the misnamed "Pascal's triangle" to evaluate $(a+b)^{10}$ then what are the first few terms?

   A. $a^{10} + 10a^9b + 11a^8b^2 + \cdots$
   B. $a^{10} + 10a^9b + 15a^8b^2 + \cdots$
   C. $a^{10} + 10a^9b + 55a^8b^2 + \cdots$
→*D*. None of the above.

---

5. How many sides would be required of a polygon if we were to use it with Archimedes' method to find π accurate to 3 places after the decimal point?

   A. 6
   B. 24
→*C*. 192
   D. None of the above.

Quiz 19

Read each question carefully.

---

1. If we represent complex numbers by pairs of real numbers then how do we multiply them?

  A. $(a, b) \cdot (c, d) = (ac, bd)$.
  B. $(a, b) \cdot (c, d) = (ad + bc, bd)$.
$\rightarrow C$. $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.
  D. None of the above.

---

2. What is Diophantos' 2-square identity?

  A. $(a^2 + b^2) \cdot (c^2 + d^2) = a^2c^2 + b^2d^2$.
  B. $(a^2 + b^2) \cdot (c^2 + d^2) = (ad + bc)^2 + b^2d^2$.
$\rightarrow C$. $(a^2 + b^2) \cdot (c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$.
  D. None of the above.

---

3. What observation of Legendre should have saved Hamilton years of fruitless search?

  A. There is a 3-square identity.
$\rightarrow B$. There can be no 3-square identity.
  C. There is a 4-square identity.
  D. There can be no 4-square identity.
  E. None of the above.

---

4. How many integral solutions are there to the equation $a^2 + b^2 + c^2 = 63$?

$\rightarrow A$. None.
  B. One.
  C. Eight.
  D. Infinitely many.
  E. None of the above.

---

5. If we assume absolute value is multiplicative, $|x| = 1$, and $x \perp 1$, then what can we say about x?

$\rightarrow A$. $x^2 = -1$.
  B. $|x + 1| = -1$.
  C. x does not exist.
  D. All of the above.
  E. None of the above.

Quiz 20

Read each question carefully.

---

1. What law of algebra did Hamilton have to give up in order to construct the quaternions?
  A. Commutativity of addition.
$\to$*B*. Commutativity of multiplication.
  C. Distributivity of multiplication over addition.
  D. All of the above.
  E. None of the above.

---

2. How many square roots of -1 are there in the quaternions?
  A. None.
  B. One.
  C. Two.
$\to$*D*. Infinitely many.
  E. None of the above.

---

3. Where were quaternions before 1843?
  A. Euler's 4-square theorem.
  B. Gauss' representation of rotations.
  C. Olinde Rodrigues' representation of rotations.
$\to$*D*. All of the above.
  E. None of the above.

---

4. Which quaternion effects the space rotation through angle θ around the axis through the unit vector $\vec{v}$?
  A. $\cos(\theta) + \vec{v}\sin(\theta)$.
  B. $\sin(\theta) + \vec{v}\cos(\theta)$.
  C. $\cos(\theta) - \vec{v}\sin(\theta)$.
$\to$*D*. None of the above.

---

5. How does a quaternion effect a space rotation?
  A. By left multiplication.
  B. By right multiplication.
$\to$*C*. By conjugation.
  D. All of the above.
  E. None of the above.

Quiz 21

Read each question carefully.

---

1. What is the climax of Euclid's *Elements*?
- A. A rigorous description of Eudoxus' theory of proportion.
- B. A complete enumeration of all pythagorean triples.
- $\to C$. Proof of the existence and uniqueness of the five regular polyhedra.
- D. None of the above.

---

2. How many rotational symmetries of the cube are there?
- A. 12
- $\to B$. 24
- C. 60
- D. None of the above.

---

3. How many rotational symmetries of the tetrahedron are there?
- $\to A$. 12
- B. 24
- C. 60
- D. None of the above.

---

4. How many regular polytopes are there in dimension 4?
- A. 3
- B. 5
- C. Infinitely many.
- $\to D$. None of the above.

---

5. How many regular polytopes are there in dimension greater than 4?
- $\to A$. 3
- B. 5
- C. Infinitely many.
- D. None of the above.

Quiz 22

Read each question carefully.

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1. Compare China before and after the Mongol conquest. Who were the rulers before and after?

Before the Genghis Khan China had fallen into disunity. The Song dynasty, especially the "Southern Song", were very dynamic, including the development of mathematics and science. After Kublai Khan's relatively short-lived Yuan dynasty collapsed the Ming came to power. This was a period of very high standards in the arts and economic development but was quite stagnant in mathematics and science.

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2. Compare the Middle East before and after the Mongol invasions. Who were dominant before and after?

Before Genghis Khan the Islamic empire had declined considerably, although it was still very powerful. Turkic groups had come to dominate, notably the Seljuqs, although they were under pressure from the Crusaders and others. Afterwards Iran and the east were dominated by the heirs of the Mongol-Turkic conquerers, whereas the west was dominated by the Ottomans. Mathematics and science stagnated throughout much of this region.

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3. Derive Gregory's series.

Gregory's series is the power series representation of the arctangent. It is derived by integrating the power series for its derivative term by term. The latter series is obtained from the geometric series.

$$\begin{aligned} \arctan(t) &= \int_0^t \frac{dt}{1+t^2} \\ &= \int_0^t (1 - t^2 + t^4 - t^6 + \cdots)\, dt \\ &= t - \tfrac{1}{3}\, t^3 + \tfrac{1}{5}\, t^5 - \tfrac{1}{7}\, t^7 + \cdots \end{aligned}$$

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4. What is the "1-2-3" identity and how does it apply to the approximation of π?

This says that $\alpha_1 = \alpha_2 + \alpha_3$ where $\alpha_n = \arctan(1/n)$. Since $\arctan(1) = \pi/4$ we can combine this identity with Gregory's series to obtain a series representation for π that converges fairly quickly:

$$\begin{aligned} \pi = &\frac{4}{2} - \frac{4}{3 \cdot 8} + \frac{4}{5 \cdot 32} - \frac{4}{7 \cdot 128} + \cdots \\ + &\frac{4}{3} - \frac{4}{3 \cdot 27} + \frac{4}{5 \cdot 243} - \frac{4}{7 \cdot 2187} + \cdots \end{aligned}$$

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5. What is Hadamard's famous aphorism? How does it apply to the approximation of π?

Hadamard said "It has been written that the shortest and best way between two truths of the real domain often passes through the imaginary one." We see this in Størmer's use of gaussian integers to generate identities that can be used with Gregory's series, as in the previous question. The point is that angles add when we multiply complex numbers. Thus since α_n is the angle of $n+i$ we can derive the "1-2-3" identity from the simple fact that

$$(2+i)(3+i) = 5 + 5i = 5 \cdot (1+i).$$

Quiz 23

Read each question carefully.

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1. According to Stillwell which of the following are discoveries, rather than inventions?
→A. Mathematical results.
B. Mathematical proofs.
C. Mathematical notation.
D. All of the above.
E. None of the above.

---

2. What is the simple fact that underlies Euclid's proof of the infinity of primes?
→A. If c divides a and b then c divides $a - b$.
B. If c divides $a - b$ then c divides a and b.
C. If c divides a and b then c divides ab.
D. If c divides ab then c divides a and b.
E. None of the above.

---

3. If a and b are integers then what is the smallest positive number of the form $ma + nb$, where m and n are integers?
→A. $\gcd(a, b)$
B. $\text{lcm}(a, b)$
C. $|a - b|$
D. $a^2 + b^2$
E. None of the above.

---

4. What is the fundamental property of prime divisors that Euclid uses in place of Unique Factorization?
→A. If p is a prime that divides a product of integers ab then either p divides a or else p divides b.
B. If p is a prime that divides a product of integers ab then $p = \gcd(a, b)$.
C. If p is a prime and $p = \gcd(a, b)$ then either p divides a or else p divides b.
D. If p is a prime and $p = \gcd(a, b)$ then either $p = a$ or else $p = b$.
E. None of the above.

---

5. What is a gaussian integer?
A. Any number of the form $a^2 + b^2$, where a and b are integers.
B. Any number of the form $a^2 - b^2$, where a and b are integers.
C. A prime number of the form $a^2 + b^2$, where a and b are integers.
D. A prime number of the form $a^2 - b^2$, where a and b are integers.
→E. None of the above.

Quiz 24

Read each question carefully.

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1. What "good luck" did Euler have, according to Stillwell?

In showing that the diophantine equation $x^3 = y^2 + 2$ has only one (positive) integer solution (namely $(3, 5)$) he used without proof the fact the the system of numbers $\{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$ enjoys the Unique Factorization Property (UFP). This is true but not obvious. For example the similar system $\{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ does *not* have the UFP

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2. What number system did Kummer prove *does not* enjoy Unique Factorization?

In refuting certain spurious proofs of Fermat's Last Theorem Kummer proved that the number system generated by the n-th roots of unity does *not* have the UFP when n is a prime and $n \geq 23$.

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3. What was Kummer's reaction to his discovery, according to Stillwell?

He refused to accept that this is the end of the story. He believed that the UFP might be restored if we augment our system with "ideal primes".

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4. What analogy does Stillwell use for Kummer's work?

Stillwell compares Kummer's work to that of nineteenth century chemists who were attempting to isolate the element fluorine. Fluorine's existence was hypothesized only through properties common to its combination with other elements.

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5. What was Kummer's basic idea, according to Stillwell?

Kummer posited that a number is known by the set of its multiples. Note the methodological link with chemists, who employ spectrography and other measures to infer the existence of elements in chemical compounds.

Quiz 25

Read each question carefully.

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1. What would a person in the 3-cylinder see while looking in the periodic direction?

The back of his or her own head, repeated infinitely often. He uses Magritte's *La reproduction interdite* to help illustrate this idea, but the difference is that unlike the painting the image of the back of your head would appear periodically, hence infinitely often.

---

2. What is one of the three ways Stillwell describes the 3-cylinder?

The 3-sphere can be described as the set of points in $\mathbb{R}^4$ at distance 1 from the (x, y)-plane; *or* as the set of points with coordinates (x, y, θ) where x and y are any real numbers and θ is any angle; *or* as the object constructed by joining opposite sides of a slab of space bounded by two parallel planes.

---

3. What is the "trick that mathematicians always use when they want two things that are not equal to be equal"?

Mathematicians would simply label them "equivalent", collect the objects into their equivalence classes, and talk about equality between equivalence classes.

---

4. What is dodecahedral space?

A finite manifold constructed by joining opposite faces of a dodecahedron. It is *not* an infinite 3-dimensional space made up of regular dodecahedra, although that is what we might see if we were standing inside it.

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5. According to Stillwell what was the starting point for Riemann's theory of elliptic functions?

Although elliptic functions get their name from the arclength integral of an ellipse, and although they are periodic, and hence amenable to Fourier analysis, they are in fact *doubly periodic*, and hence can be thought of as functions on a 2-torus, the way trigonometric functions can be defined as functions on the circle. This was Riemann's approach.

Quiz 26

Read each question carefully.

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1. How does Stillwell adapt Lucretius' argument in the context of numbers?

Lucretius argued that space is infinite, wondering what would happen if you came to the edge and threw a javelin. Stillwell argues that we expect that the natural numbers are infinite, wondering what would happen if you came to the last number and added one.

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2. In what way do countable sets fit nicely with Gauss' injunction against infinity?

Countable sets can be enumerated. At any point in the enumeration you have only named finitely many numbers. Eventually each number in an infinite set is named in an enumeration, but you never have to refer to infinitely many.

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3. What is the difference between a countable and an uncountable set?

An uncountable set cannot be enumerated — that is, there is no bijection between an uncountable set and the set of natural numbers. Thus we would expect that some properties of uncountable sets are revealed only by reference to infinitely many elements at a time.

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4. What does Cantor's "diagonal argument" prove?

Cantor's diagonal argument is one way to prove that the set of real numbers is uncountable.

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5. What is a transcendental number?

An algebraic number is the root of some polynomial having rational coefficients. One example is $\sqrt{2}$. A transcendental number is one that is not algebraic. One example is π. Cantor showed that the set of algebraic numbers is countable, and hence that the set of transcendental numbers is uncountable.