Math 1270-011, Summer 09: Answers to quiz 1

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1. Evaluate $\lim_{x\to -2} \frac{x^2-x-6}{x+2}$.

$$\lim_{x\to -2} \frac{x^2-x-6}{x+2} = \lim_{x\to -2} \frac{(x+2)(x-3)}{x+2} = \lim_{x\to -2}(x-3) = -2-3 = -5.$$

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2. Use the definition of derivative to find $\frac{df}{dx}$, where $f(x) = 2x^3+5$.

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h\to 0} \frac{(2(x+h)^3+5)-(2x^3+5)}{h} \\
&= \lim_{h\to 0} \frac{2x^3+6x^2h+6xh^2+2h^3+5-2x^3-5}{h} \\
&= \lim_{h\to 0} \frac{(6x^2+6xh+2h^2)h}{h} \\
&= \lim_{h\to 0}(6x^2+6xh+2h^2) \\
&= 6x^2+0+0 = 6x^2.
\end{aligned}$$

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3. Find an equation for tangent line to the graph of $y = x^4-5x^3+2$ at the point $x = 2$.

Since $\frac{dy}{dx} = 4x^3-15x^2$, the slope of the tangent line is $4\cdot 2^3-15\cdot 2^2 = -28$. Since $y(2) = 2^4-5\cdot 2^3+2 = -22$ the point-slope form of the equation is $y+22 = -28(x-2)$.

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4. Find the derivative of $y = \frac{t^2e^{2t}}{t+e^{3t}}$.

$$\frac{dy}{dx} = \frac{(t+e^{3t})(2t\cdot e^{2t}+t^2\cdot 2e^{2t})-(t^2e^{2t})(1+3e^{3t})}{(t+e^{3t})^2}.$$

Math 1270-011, Summer 09: Answers to quiz 2

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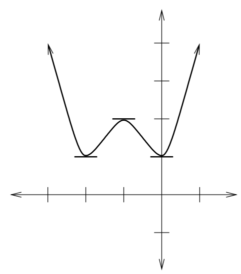
1. Find the critical numbers for $f(x) = x^4 + 4x^3 + 4x^2 + 1$, then find the intervals on which f is increasing and the intervals on which it is decreasing.

$$\frac{df}{dx} = 4x^3 + 12x^2 + 8x = 4x(x+1)(x+2).$$

Since the derivative is defined everywhere, the critical numbers are those for which the derivative is 0: namely $x = 0$, $x = -1$, and $x = -2$. At these points, f takes the values 1, 2 and 1, respectively. Since the dominant term is $4x^4$, $\lim_{x\to\pm\infty} f(x) = +\infty$. In summary:

x	$-\infty$	-2	-1	0	$+\infty$
f	$+\infty$	1	2	1	$+\infty$
f'		0	0	0	

Hence f decreases from $+\infty$ to 1 on the interval $(-\infty, 2)$; increases from 1 to 2 on the interval $(-2, -1)$; decreases from 2 back to 1 on the interval $(-1, 0)$; and finally increases from 1 to $+\infty$ on the interval $(0, +\infty)$.



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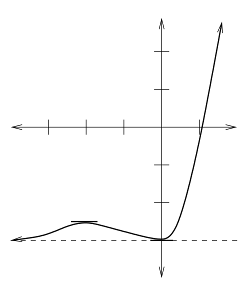
2. Find all of the relative extrema for $f(x) = x^2 e^x - 3$.

$$\frac{df}{dx} = 2xe^x + x^2 e^x = x(x+2)e^x.$$

Since the derivative is defined everywhere, the critical numbers are those for which the derivative is 0: namely $x = 0$ and $x = -2$. At these points, f takes the values -3 and $4e^{-2} - 3 \approx -2.46$, respectively. Since e^x dominates the factor x^2, $\lim_{x\to-\infty} f(x) = 0 - 3 = -3$ and $\lim_{x\to+\infty} f(x) = +\infty$. In summary:

x	-3	-2	0	$+\infty$
f	0	-2.46	-3	$+\infty$
f'		0	0	

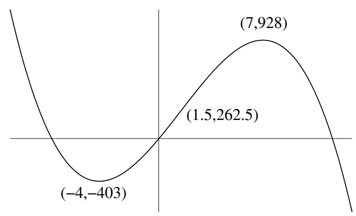
Hence f increases from -3 to -2.46 (approximately) on the interval $(-\infty, -2)$; decreases from -2.46 (approximately) to -3 on the interval $(-2, 0)$; and finally increases from -3 to $+\infty$ on the interval $(0, +\infty)$. Hence f has a relative maximum at $x = -2$ and a relative minimum at $x = 0$.



3. Find the intervals on which $f(x) = -2x^3 + 9x^2 + 168x - 3$ is concave up, the intervals on which it is concave down, and any inflection points.

$$\frac{df}{dx} = -6x^2 + 18x + 168 = -6(x+4)(x-7), \qquad \frac{d^2 f}{dx^2} = -12x + 18 = -6(2x-3).$$

The second derivative is 0 when $x = \frac{3}{2}$. On the interval $(-\infty, \frac{3}{2})$ the second derivative is positive, and hence the graph of f is concave up. On the interval $(\frac{3}{2}, +\infty)$ the second derivative is negative, and hence the graph of f is concave down. Hence the one and only inflection point is at $x = \frac{3}{2}$.



4. Graph the function $f(x) = \dfrac{1}{x^2 - 9}$.

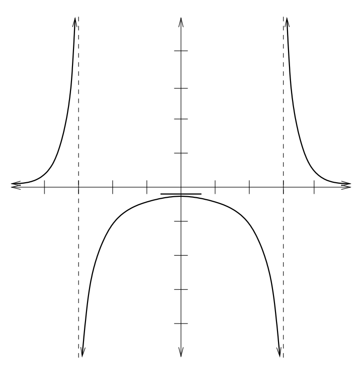
The function has vertical asymptotes at $x = \pm 3$, and tends to 0 as $x \to \pm\infty$. The function is negative when $|x| < 3$ and positive otherwise.

$$\frac{df}{dx} = \frac{d}{dx}\left[(x^2-9)^{-1}\right] = -(x^2-9)^{-2} \cdot 2x = \frac{-2x}{(x^2-9)^2}.$$

Since the denominator is always positive except when $x = \pm 3$, the slope is positive when $x < 0$ and $x \neq -3$; and the slope is negative when $x > 0$ and $x \neq 3$. The only critical number in the domain of f is $x = 0$, at which f takes the value $-\frac{1}{9}$. In summary:

x	$-\infty$	-3^-	-3^+	0	3^-	3^+	$+\infty$
f	0	$+\infty$	$-\infty$	$-\frac{1}{9}$	$-\infty$	$+\infty$	0
f'				0			

Hence f increases from 0 to $+\infty$ on the interval $(-\infty, -3)$; decreases from $+\infty$ to $-\frac{1}{9}$ on the interval $(-3, 0)$; increases from $-\frac{1}{9}$ to $+\infty$ on the interval $(0, 3)$; and finally decreases from $+\infty$ to 0 on the interval $(3, +\infty)$.



Math 1270-011, Summer 09: Answers to quiz 3

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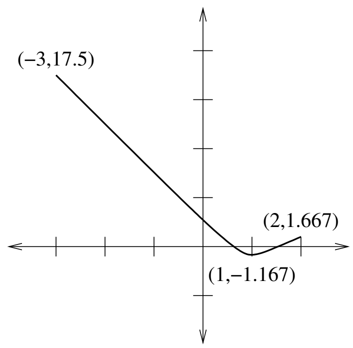
1. Find the absolute extrema for $f(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 - 4x + 1$ on the interval $[-3, 2]$.

$$\frac{df}{dx} = x^3 + 3x - 4 = (x+4)(x-1).$$

There are two zeroes of this expression but only one gives a critical point in the given interval, namely $x = 1$. Our table contains three points:

x :	-3	1	2
f :	17.5	-1.167	1.667
f' :		0	

Hence the maximum is 17.5, which occurs at $x = -3$; and the minimum is (approximately) -1.167, which occurs at $x = 1$.



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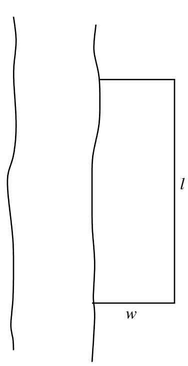
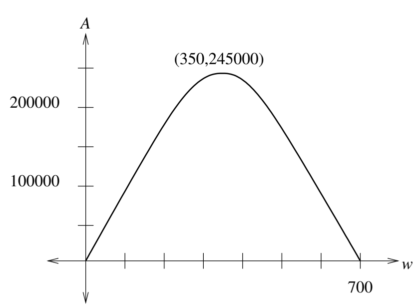
2. A campground owner has 1400 m of fencing. He wants to enclose a rectangular field bordering a river, with no fencing needed along the river. Find the maximum area and the dimensions which achieve this maximum.

If the width is w and the length is ℓ then $1400 = 2w + \ell$ and the area A is given by the formula

$$A = A(w) = w\ell = w(1400 - 2w) = 1400w - 2w^2.$$

Since $w \geq 0$ and $\ell = 1400 - 2w \geq 0$ we must restrict w to the interval $[0, 700]$. Now $\frac{dA}{dw} = 1400 - 4w$, which is 0 when $w = 350$. Since $A = 0$ at the endpoints $w = 0$ and $w = 700$ we conclude that the largest such field is 350×700.





3. A manufacturer has a steady annual demand for 13,950 cases of sugar. It costs \$9 to store 1 case for 1 year, \$31 in setup costs to produce each batch, and \$16 to produce each case. Find the number of cases per batch that should be produced.

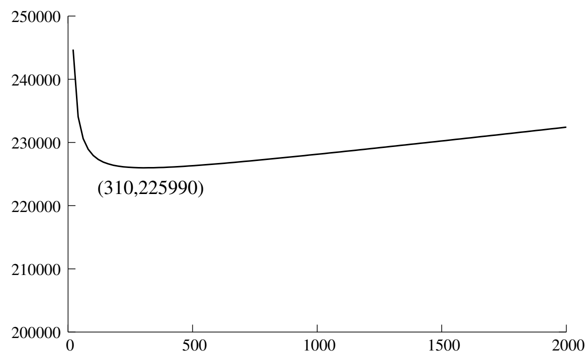
If q is the number of cases per batch then $13950/q$ is the number of batches per yer, and $q/2$ is the average inventory. Each batch costs $31 + 16q$ dollars to produce. Hence the total cost C is

$$\begin{aligned} C(q) &= \text{storage costs} + \text{producton costs} \\ &= 9q/2 + (31 + 16q) \cdot 13950/q \\ &= 4.5q + 432450q^{-1} + 223200. \end{aligned}$$

We should choose the batch size so as to minimize total costs. Since C tends to infinity when q tends to 0 and also when q tends to infinity, this minimum occurs when the marginal cost is 0:

$$\frac{dC}{dq} = 4.5 - 432450q^{-2} = 0 \implies q = \sqrt{432450/4.5} = 310.$$

Hence the batch size should be 310, for a total of 45 batches at a total cost of \$225990.



4. Find $\dfrac{dy}{dx}$ where $5x^3 = 3y^2 + 4y$.

$$15x^2 = 6y\,\frac{dy}{dx} + 4\frac{dy}{dx} = (6y + 4)\,\frac{dy}{dx}.$$

Hence

$$\frac{dy}{dx} = \frac{15x^2}{6y + 4}.$$

Math 1270-011, Summer 09: Answers to quiz 4

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1. Evaluate $\int \frac{6t-\sqrt{t}}{3t}\,dt$.

$$\int \frac{6t-\sqrt{t}}{3t}\,dt = \int (2 - \tfrac{1}{3}t^{-1/2})\,dt = 2t - \tfrac{1}{3}\cdot\frac{t^{1/2}}{1/2} + C = 2t - \tfrac{2}{3}t^{1/2} + C.$$

Check:

$$\frac{d}{dt}\left(2t - \tfrac{2}{3}t^{1/2} + C\right) = 2 - \tfrac{2}{3}\cdot\tfrac{1}{2}t^{-1/2} = \frac{3\cdot 3t}{3t} - \frac{t^{1/2}}{3t} = \frac{6t-\sqrt{t}}{3t}.$$

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2. Evaluate $\int \frac{\sqrt{5+\ln|t|}}{2t}\,dt$.

If we let $u = 5+\ln|t|$ then $du = \frac{1}{t}\,dt$ and

$$\int \frac{\sqrt{5+\ln|t|}}{2t}\,dt = \tfrac{1}{2}\int \sqrt{u}\,du = \tfrac{1}{2}\int u^{1/2}\,du = \tfrac{1}{2}\cdot\tfrac{2}{3}u^{3/2} + C = \tfrac{1}{3}(5+\ln|t|)^{3/2} + C.$$

Check:

$$\frac{d}{dt}\left(\tfrac{1}{3}(5+\ln|t|)^{3/2} + C\right) = \tfrac{1}{3}\cdot\tfrac{3}{2}(5+\ln|t|)^{1/2}\cdot\frac{1}{t} = \frac{\sqrt{5+\ln|t|}}{2t}.$$

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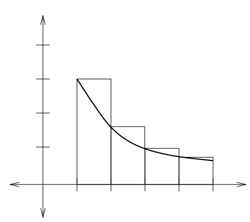
3. Approximate $\int_1^5 \frac{3}{t}\,dt$ using left endpoints with $n=4$.

The width of each rectangle is $\frac{1}{4}(5-1)=1$, hence

$$\int_1^5 \frac{3}{t}\,dt \approx \tfrac{3}{1}\cdot 1 + \tfrac{3}{2}\cdot 1 + \tfrac{3}{3}\cdot 1 + \tfrac{3}{4}\cdot 1 = \tfrac{25}{16} = 6.25.$$



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4. Evaluate $\int_1^3 e^{4t}\,dt$.

To find the antiderivative of e^{4t} we use the substitution $u=4t$, $du=4dt$. When $t=1$, $u=4$; and when $t=3$, $u=12$. Hence

$$\int_1^3 e^{4t}\,dt = \tfrac{1}{4}\int_3^{12} e^u\,du = \tfrac{1}{4}e^u\Big|_3^{12} = \tfrac{1}{4}(e^{12}-e^3) \approx 40683.67.$$

Math 1270-011, Summer 09: Answers to quiz 5

If $f(x)$ is the rate of money flow at an annual interest rate of r over t years then

$$\text{total money flow} = \int_0^t f(x)\,dx$$
$$\text{present value} = \int_0^t f(x)e^{-rx}\,dx$$
$$\text{accumulated amount} = e^{rt}\int_0^t f(x)e^{-rx}\,dx$$

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1. Evaluate $\int (4x-12)e^{-8x}\,dx$.

If we let $A = 4x - 12$ and $B' = e^{-8x}$ then $A' = 4$, $B = -\frac{1}{8}e^{-8x}$ and

$$\begin{aligned}\int (4x-12)e^{-8x}\,dx &= (4x-12)(-\frac{1}{8}e^{-8x}) - \int (-\frac{4}{8}e^{-8x})\,dx \\ &= \frac{1}{8}(12-4x)e^{-8x} - \frac{1}{16}e^{-8x} + C \\ &= \frac{1}{16}(23-8x)e^{-8x} + C.\end{aligned}$$

Check:

$$\frac{d}{dx}\left(\frac{1}{16}(23-4x)e^{-8x}+C\right) = \frac{1}{16}\big((-8)e^{-8x} + (23-8x)e^{-8x}\cdot(-8)\big) = (4x-12)e^{-8x}.$$

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2. Find the average value of $f(x) = 3x^2 - 4$ on the interval $[1, 5]$.

$$\text{average value} = \frac{1}{5-1}\int_1^5 (3x^2-4)\,dx = \frac{1}{4}(x^3-4x)\Big|_1^5 = \frac{1}{4}(105 - -3) = 27.$$

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3. The rate of a continuous flow of money starts at \$5000 and decreases exponentially at a rate of 1% per year for 8 years. Find the present value and final amount at an annual interest rate of 8% compounded continuously.

If $f(x)$ is the rate of money flow, then $f(x) = 5000e^{-kx}$ where

$$e^{-k\cdot 1} = 0.99, \text{ hence } k = -\ln(0.99) \approx 0.01.$$

Alternatively,

$$-0.01 = \frac{1}{f(x)}\frac{df}{dx} = -k.$$

Hence if P denotes the present value and A the accumulated amount after 8 years then

$$P = \int_0^8 5000e^{-0.01x}e^{-0.08x}\,dx = -\frac{5000}{0.09}e^{-0.09x}\Big|_0^8 = 28513.76$$

and $A = e^{0.08\cdot 8}P = 54075.81$.

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4. Determine whether or not the integral $\int_0^\infty 8e^{-8x}\,dx$ converges, and determine its value if it does.

$$\int_0^\infty 8e^{-8x}\,dx = \lim_{t\to\infty}\int_0^t 8e^{-8x}\,dx = \lim_{t\to\infty} -e^{-8x}\Big|_0^t = \lim_{t\to\infty}(1-e^{-8t}) = 1 - 0 = 1.$$

Hence the integral converges, and the value is 1.

Math 1270-011, Summer 09: Answers to quiz 6

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1. Use the echelon method to solve the system

$$\begin{array}{rcrcl} x & + & y & = & 5 \\ 2x & - & 2y & = & 2 \end{array}$$

Label each row operation!

$$\begin{array}{rcrcrl} x & + & y & = & 5 & \\ & & -4y & = & -8 & (R_2 - 2R_1 \to R_2) \end{array}$$

$$\begin{array}{rcrcrl} x & + & y & = & 5 & \\ & & y & = & 2 & (-\frac{1}{4}R_2 \to R_2) \end{array}$$

This is now in echelon form, so we can apply back substitution:

$$y = 2$$

$$x + 2 = 5 \implies x = 3$$

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2. Use the Gauss-Jordan method to solve the system

$$\begin{array}{rcrcrcrcr} x & + & y & - & z & + & 2w & = & -20 \\ 2x & - & y & + & z & + & w & = & 11 \\ 3x & - & 2y & + & z & - & 2w & = & 27 \end{array}$$

Label each row operation!

$$\begin{array}{rrrr|r} 1 & 1 & -1 & 2 & -20 \\ 2 & -1 & 1 & 1 & 11 \\ 3 & -2 & 1 & -2 & 27 \end{array}$$

$$\begin{array}{rrrr|rl} 1 & 1 & -1 & 2 & -20 & \\ 0 & -3 & 3 & -3 & 51 & (R_2 - 2R_1 \to R_2) \\ 0 & -5 & 4 & -8 & 87 & (R_3 - 3R_1 \to R_3) \end{array}$$

$$\begin{array}{rrrr|rl} 1 & 1 & -1 & 2 & -20 & \\ 0 & 1 & -1 & 1 & -17 & (-\frac{1}{3}R_2 \to R_2) \\ 0 & -5 & 4 & -8 & 87 & \end{array}$$

$$\begin{array}{rrrr|rl} 1 & 0 & 0 & 1 & -3 & (R_1 - R_2 \to R_2) \\ 0 & 1 & -1 & 1 & -17 & \\ 0 & 0 & -1 & -3 & 2 & (R_3 + 5R_2 \to R_3) \end{array}$$

$$\begin{array}{rrrr|rl} 1 & 0 & 0 & 1 & -3 & \\ 0 & 1 & -1 & 1 & -17 & \\ 0 & 0 & 1 & 3 & -2 & (-R_3 \to R_3) \end{array}$$

$$\begin{array}{rrrr|rl} 1 & 0 & 0 & 1 & -3 & \\ 0 & 1 & 0 & 4 & -19 & (R_2 + R_3 \to R_2) \\ 0 & 0 & 1 & 3 & -2 & \end{array}$$

The augmented coefficient matrix is now in reduced row echelon form. From this form we see that w is a free variable, and that the final solution is $\{x = -w - 3, y = -4w - 19, z = -3w - 2, w\}$.

3. Perform the given operations, where possible. If it is not possible, explain why not.

(a) $\begin{bmatrix} 1 & 3 & -2 \\ 4 & 7 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 6 & 4 \\ -5 & 2 \end{bmatrix}$

Addition is not possible in this instance, since the matrices are of different sizes.

(b) $\begin{bmatrix} 2 & 3 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

$$\begin{bmatrix} 2+4-3 & 3+3-2 \\ -2+7-1 & 4+8-4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 8 \end{bmatrix}$$

4. Perform the given operations, where possible. If it is not possible, explain why not.

(a) $\begin{bmatrix} 2 & -1 & 7 \\ -3 & 0 & -4 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 2 \cdot 5 - 1 \cdot 10 + 7 \cdot 2 \\ -3 \cdot 5 + 0 \cdot 10 - 4 \cdot 2 \end{bmatrix} = \begin{bmatrix} 14 \\ -23 \end{bmatrix}$$

(b) $\begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} -1 & 0 & 4 \\ 5 & -2 & 0 \end{bmatrix}$

$$\begin{bmatrix} 2 \cdot (-1) - 1 \cdot 5 & 2 \cdot 0 - 1 \cdot (-2) & 2 \cdot 4 - 1 \cdot 0 \\ 3 \cdot (-1) + 6 \cdot 5 & 3 \cdot 0 + 6 \cdot (-2) & 3 \cdot 4 + 6 \cdot 0 \end{bmatrix} = \begin{bmatrix} -7 & 2 & 8 \\ 27 & -12 & 12 \end{bmatrix}$$

Math 1270-011, Summer 09: Answers to quiz 7

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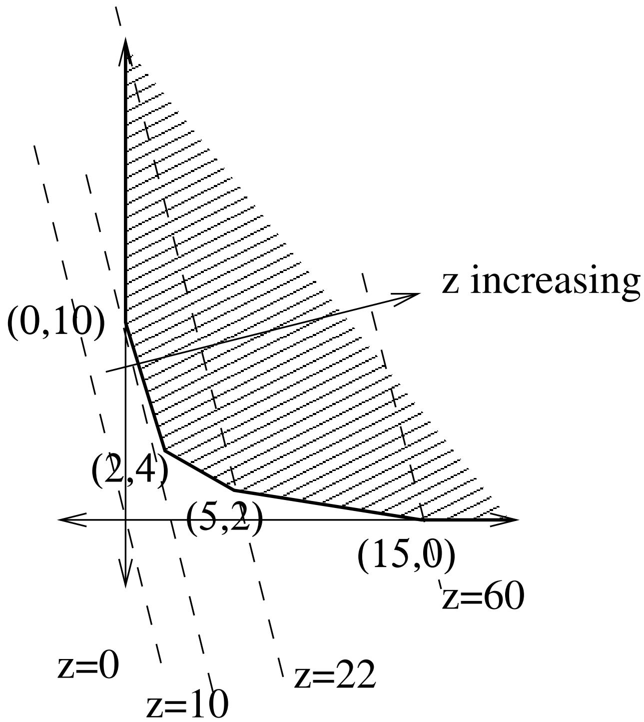
1. Find the maximum and minimum values (if they exist) of $z = 4x + y$ on the region pictured below. You must justify your assertions!



We begin with a table of the z-values at the corner points.

x	:	0	2	5	15
y	:	10	4	2	0
$z = 4x + y$	:	10	12	22	60

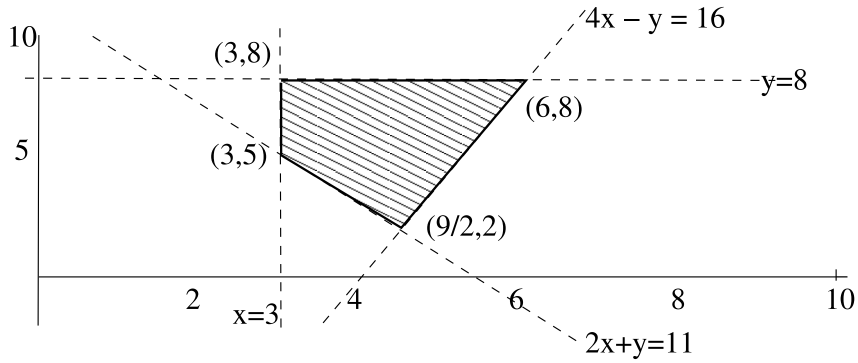
Since the region is unbounded we must look at the lines $z = 0$, $z = 10$, etc. We see that z increases without bound in the feasible region, and hence there is no maximum. However, z does have a minimum in the feasible region, since the line $z = 0$ misses the region entirely. By the Corner Point Theorem, the minimum occurs at a corner point. In our case, the minimum is 10, which occurs at the point $(0, 10)$.

2. Find the maximum and minimum values (if they exist) of $z = 5x+2y$ subject to the following constraints:

$$\begin{aligned} 4x - y &\le 16 \\ 2x + y &\ge 11 \\ x &\ge 3 \\ y &\le 8 \end{aligned}$$

Draw a picture, and justify your assertions!

We first graph the boundary lines $4x - y = 16$, $2x + y = 11$, $x = 3$, and $y = 8$:



The origin $(0, 0)$ satisfies the first and last inequalities, but not the middle two. Hence the feasible region lies to the left of the first boundary line; above the second; to the right of the third; and below the last. This maps out a bounded feasible region, as shaded above.

To map out the feasible region more precisely, we determine the four corner points. The only nontrivial one is the solution to the system

$$\begin{aligned} 4x - y &= 16 \\ 2x + y &= 11 \end{aligned}$$

If we apply the echelon method and back substitution we find that $x = 9/2$, $y = 2$. The remaining three corner points can be obtained by setting either $x = 3$ or $y = 8$ into one of the other equations.

Since the region is bounded and z is continuous, z attains both a maximum and a minimum on the region. By the Corner Point Theorem, these optima must occur at one of the four corner points. Here is a table of z values:

x	:	3	3	6	9/2
y	:	5	8	8	2
$z = 5x + 2y$	:	25	31	46	53/2

We conclude that the maximum is 46, attained at $(6, 8)$; and the minimum is 25, attained at $(3, 5)$.