Math 4/5880, Spring 2015: Quiz Answers

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Quiz 1

Give *brief* but complete answers to the following.

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1. When and where were complex numbers born?

Complex numbers were born in Renaissance Italy. Specifically they were first taken seriously by the Italian engineer Bombelli in 1572, in his analysis of the so-called *casus irreducibilis* in Cardano's formula for the cubic.

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2. What is the modulus of a complex number?

If z is a complex number then its modulus, denoted $|z|$, is the distance from z to the origin. If $z = x + iy$, where x and y are real, then $|z| = \sqrt{x^2 + y^2}$. If $z = re^{i\theta}$, where $r > 0$ and θ is real then $|z| = r$.

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3. What is the Triangle Inequality?

If z and w are complex numbers then the Triangle Inequality says that $||z| - |w|| \le |z + w| \le |z| + |w|$. It is called the Triangle Inequality because z, w, and $z + w$ can be regarded as the sides of a triangle, and the double inequality above is satisfied by the sides of any triangle.

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4. What is the principal value of the argument of a complex number?

If $z = re^{i\theta}$, where $r > 0$ and θ is real then θ is referred to as an *argument* for z. Of course for a given z there are infinitely many choices for its polar coordinate θ, all differing by an integral multiple of 2π. The *principal* argument is the unique one of these satisfying $-\pi < \theta \le \pi$.

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5. What is Euler's Formula?

Euler's formula says that if θ is real then $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. (Actually, we will see later that this identity is true for all θ, real or complex.)

Quiz 2

Give *brief* but complete answers to the following.

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1. What does it mean when we write $f(z) = O(g(z))$ as $z \to \infty$.
This means that there are positive constants M and C such that

$$|z| > M \implies |f(z)| \leq C|g(z)|.$$

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2. What does it mean when we write $f(z) = o(g(z))$ as $z \to 0$.
This means that

$$\lim_{z \to 0} \frac{f(z)}{g(z)} = 0.$$

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3. What are the Cauchy-Riemann equations?
If u and v are the real and imaginary parts of the complex differentiable function f then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

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4. How does our text define e^z, $\sin(z)$, and $\cos(z)$?
These are defined by the familiar power series from Calc II:

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \\ \sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \\ \cos(z) &= 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \end{aligned}$$

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5. Is it true that $\log(wz) = \log(w) + \log(z)$ for all nonzero w and z? Explain!
This is false, in general, for example $\log(-1) = \pi i$ but

$$\log((-1)^2)) = \log(1) = 0 \neq 2\log(-1) = 2\pi i.$$

Quiz 3

Give *brief* but complete answers to the following.

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1. Using our text's definitions of e^z, $\sin(z)$, and $\cos(z)$, verify that

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \ \cos(z) = \frac{e^{iz} + e^{-iz}}{2}.$$

$$\begin{aligned}
e^{iz} - e^{-iz} &= 1 + iz - \frac{z^2}{2} - i\frac{z^3}{3!} + \frac{z^4}{4!} + \cdots - 1 + iz + \frac{z^2}{2} - i\frac{z^3}{3!} - \frac{z^4}{4!} + \cdots \\
&= 2i \cdot (z - \frac{z^3}{3!} + \cdots) = 2i\sin(z) \\
e^{iz} + e^{-iz} &= 1 + iz - \frac{z^2}{2} - i\frac{z^3}{3!} + \frac{z^4}{4!} + \cdots + 1 - iz - \frac{z^2}{2} + i\frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \\
&= 2 \cdot (1 - \frac{z^2}{4!} + \frac{z^4}{4!} - \cdots) = 2\cos(z)
\end{aligned}$$

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2. Using the formulas in problem 1 verify that $\sin^2(z) + \cos^2(z) = 1$.

$$\sin^2(z) + \cos^2(z) = -\frac{e^{2iz} - 2 + e^{-2iz}}{4} + \frac{e^{2iz} + 2 + e^{-2iz}}{4} = \frac{4}{4} = 1.$$

Alternatively:

$$\frac{d}{dz}(\sin^2(z) + \cos^2(z)) = 2\sin(z)\cos(z) + 2\cos(z)(-\sin(z)) = 0.$$

Hence $\sin^2(z) + \cos^2(z) = \sin^2(0) + \cos^2(0) = 1$ for all z.

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3. Verify that $\sin(iz) = i\sinh(z)$ and $\cos(iz) = \cosh(z)$.

$$\begin{aligned}
\sin(iz) &= \frac{e^{-z} - e^{z}}{2i} = i\frac{e^{z} - e^{-z}}{2} = i\sinh(z) \\
\cos(iz) &= \frac{e^{-z} + e^{z}}{2} = \frac{e^{z} + e^{-z}}{2} = \cosh(z)
\end{aligned}$$

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4. Use the addition formulas for the trig functions and any of the formulas above to prove that if x and y are the real and imaginary parts of z then

$$|\sin(z)|^2 = \sin^2(x) + \sinh^2(y), \ |\cos(z)|^2 = \cos^2(x) + \sinh^2(y).$$

$$\begin{aligned}
|\sin(x + iy)|^2 &= |\sin(x)\cos(iy) + \cos(x)\sin(iy)|^2 \\
&= |\sin(x)\cosh(y) + i\cos(x)\sinh(y)|^2 \\
&= \sin^2(x)\cosh^2(y) + \cos^2(x)\sinh^2(y) \\
&= \sin^2(x)(1 + \sinh^2(y)) + \cos^2(x)\sinh^2(y) \\
&= \sin^2(x) + \sinh^2(y).
\end{aligned}$$

For the second identity we could either proceed similarly or we could use the fact that $\cos(z) = \sin(\frac{1}{2}\pi - z)$.

Quiz 4

Give *brief* but complete answers to the following.

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1. What is a smooth path?

A path is a continuous function γ from some real interval $[a, b]$ into the complex plane $\mathbb{C}$. The path is smooth if it has a continuous derivative $d\gamma/dt$.

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2. What is a piece-wise smooth path?

A path $\gamma\colon [a, b] \to \mathbb{C}$ is piece-wise smooth if the domain $[a, b]$ can be partitioned into finitely many subintervals on each of which γ is smooth.

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3. If γ is a path then what are γ^* and $\overleftarrow{\gamma}$?

γ^* is the image of γ. That is,

$$\gamma^* = \{\gamma(t) : t \in [a, b]\}.$$

$\overleftarrow{\gamma}$ is a path that traverses γ^* in reverse. More precisely,

$$\overleftarrow{\gamma}(t) = \gamma(b + a - t).$$

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4. What is a closed path? What is a simple path?

A closed path is one that begins and ends at the same point: $\gamma(a) = \gamma(b)$. A simple path is one that never crosses itself — except possibly to close up. More precisely, if $a \leq t_1 < t_2 \leq b$ then $\gamma(t_1) \neq \gamma(t_2)$ — except possibly $\gamma(a) = \gamma(b)$.

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5. If γ is a piece-wise smooth path and f is a continuous function whose domain contains γ^* then what do we mean by $\int_\gamma f(z)\,dz$?

$$\int_\gamma f(z)\,dz = \int_a^b \mathrm{Re}(f(\gamma(t)\frac{d\gamma}{dt})\,dt + i\int_a^b \mathrm{Im}(f(\gamma(t)\frac{d\gamma}{dt})\,dt$$

Alternatively, if u and v are the real and imaginary parts of f and if x and y are the real and imaginary parts of γ then

$$\int_\gamma f(z)\,dz = \int_a^b [u(\gamma(t))\frac{dx}{dt} - v(\gamma(t))\frac{dy}{dt}]\,dt = i\int_a^b [u(\gamma(t))\frac{dy}{dt} + v(\gamma(t))\frac{dx}{dt}]\,dt.$$

Often this form of the integral is abbreviated

$$\int_\gamma f(z)\,dz = \int_\gamma u\,dx - v\,dy + i\int_\gamma u\,dy + v\,dx.$$

Quiz 5

Give *brief* but complete answers to the following.

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1. What is the Cauchy-Goursat Theorem? Be precise!

This theorem says that if f is holomorphic in a domain that includes a piece-wise smooth simple closed contour C and its interior then $\int_C f = 0$.

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2. What were the individual contributions of Cauchy and Goursat to their joint theorem?

Cauchy first proved this theorem under the additional assumption that f' is continuous. His proof was an application of Green's Theorem to the real and imaginary parts of the integral. Goursat found a way to avoid the hypothesis that f' be continuous. He used the antiderivative theorem (the fundamental theorem for path integrals) to reduce to the case that C is a triangle, and then used a clever argument of repeatedly subdividing the triangle into smaller triangles.

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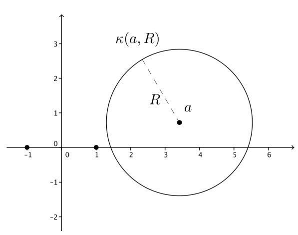
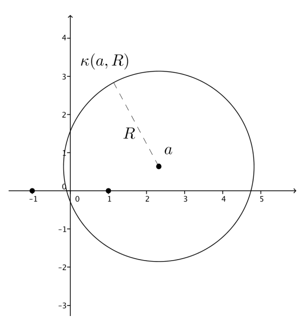
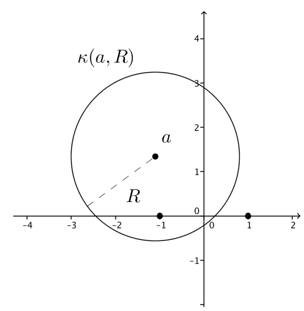
3. Discuss the various cases you must consider if asked to evaluate $\oint_{\kappa(a,R)} \frac{dz}{z^2-1}$. Illustrate each case and sketch how you might evaluate the integral in that case.

If either or both of ± 1 lies on the circle $\kappa(a,R)$ then the integral is not defined. If neither ± 1 are inside the circle then the integral is 0, by the Cauchy-Goursat Theorem. If one or both of ± 1 is inside the circle then there are at least two ways to proceed.





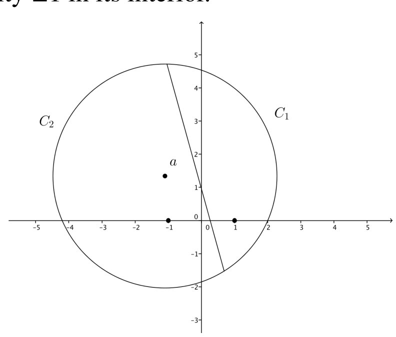


First, we can use the method of "partial fractions" to rewrite the integral as

$$\oint_{\kappa(a,R)} \frac{dz}{z^2-1} = \oint_{\kappa(a,R)} \frac{\frac{1}{2}\,dz}{z-1} - \oint_{\kappa(a,R)} \frac{\frac{1}{2}\,dz}{z+1}.$$

Now we can evaluate each integral separately, using the Cauchy Integral Formula; or else using the Cauchy-Goursat Theorem to reduce to the case of a small circle centered at ± 1, together with an explicit parametriztion of the circle.

Alternatively we could, if necessary, divide the circle by a chord, so that we replace $\kappa(a,R)$ by a union of contours $C_1 + C_2$ each of which has one singularity ± 1 in its interior.



We can then apply the Cauchy Integral Formula to each integral.

$$\oint_{\kappa(a,R)} \frac{dz}{z^2-1} = \oint_{C_1} \frac{(z+1)^{-1}\,dz}{z-1} + \oint_{C_2} \frac{(z-1)^{-1}\,dz}{z+1}.$$

Quiz 6

Give *brief* but complete answers to the following.
Provide definitions and an example of each of the following.

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1. The principal part of a Laurent series.

This is the sum of the terms $a_n(z-c)^n$ in the Laurent that have negative exponent n. For $|z| < 1$ we have the Laurent

$$\frac{1}{z^3 + z^2} = \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 - \cdots$$

Its principal part is $z^{-2} - z^{-1}$.

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2. A pole.

This is an isolated singularity c for a function f such that for some positive integer n we have a finite nonzero limit

$$\lim_{z \to c} (z-c)^n f(z).$$

A pole can be recognized as an isolated singularity whose principal part is nonzero but has only finitely many terms. Alternatively, f has a pole at c if

$$f(z) = (z-c)^{-n} \phi(z)$$

where $n \geq 1$ and ϕ is holomorphic and nonzero in a deleted neighborhood of c.

Thus 0 is an isolated singularity for the example in problem 2. Indeed

$$\lim_{z \to 0} z^2 \cdot \frac{1}{z^3 + z^2} = 1.$$

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3. $\mathrm{ord}(f, c)$

If the Laurent series for f in a deleted neighborhood of a point c is

$$\sum_{n=-\infty}^{\infty} a_n (z-c)^n$$

then

$$\mathrm{ord}(f, c) = \inf\{n \colon a_n \neq 0\}.$$

Said another way, if f has an (isolated) essential singularity at c then $\mathrm{ord}(f, c) = -\infty$. If f is identically 0 then $\mathrm{ord}(f, c) = +\infty$. In all other cases $\mathrm{ord}(f, c)$ is the unique integer n (positive, negative, or zero) such that the limit

$$\lim_{z - c} (z-c)^n f(z)$$

exists and is finite and nonzero. Alternatively, if

$$f(z) = (z-c)^n \phi(z)$$

where n is an integer and ϕ is holomorphic in a deleted neighborhood of c then $\mathrm{ord}(f, c) = n$.

So, in the example from problem 2 we see that

$$\mathrm{ord}\left(\frac{1}{z^3 + z^2}, 0\right) = -2.$$

4. res(f, c)

Using the same notation from the previous problem we define res$(f, c) = a_{-1}$. So, in the example from problem 2 we see that

$$\operatorname{res}\left(\frac{1}{z^3 + z^2}, 0\right) = -1.$$

5. A meromorphic function.

If all the singularities of f in a domain D are (at worst) poles then f is meromorphic in D. In the example from problem 2

$$f(z) = \frac{1}{z^3 + z^2}$$

is meromorphic in the entire complex plane. We have already seen that it has a pole of order 2 at 0. The only other singularity is at -1. At this point we find, either from the factorization

$$\frac{1}{z^3 + z^2} = (z + 1)^{-1} \cdot \frac{1}{z^2}$$

or from the limit

$$\lim_{z \to -1} (z + 1) \cdot \frac{1}{z^3 + z^2} = 1$$

that f has a simple pole at -1.

Quiz 7

Give *brief* but complete answers to the following.
Evaluate two of the following three integrals.

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1. $\displaystyle\int_0^\infty \frac{x^2\,dx}{1+x^4}$

Let $f(z) = z^2/(1+z^4)$. This function is meromorphic with exactly four simple poles, namely $e^{i(2k+1)\pi/4}$. If $|z| = R$ and R is sufficiently large then $|f(z)| \le R^2/(R^4-1) = O(1/R^2)$ as $R \to \infty$. In particular the improper integral converges. Hence we may apply Theorem 9.1 to conclude that

$$\begin{aligned}\int_0^\infty \frac{x^2\,dx}{1+x^4} &= \frac{1}{2}\cdot 2\pi i \cdot \left(\operatorname{res}(f, e^{i\pi/4}) + \operatorname{res}(f, e^{i3\pi/4})\right) \\ &= \frac{\pi i}{4}(e^{-i\pi/4} + e^{-i3\pi/4}) = \frac{\pi}{2\sqrt{2}}.\end{aligned}$$

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2. $\displaystyle\int_{-\infty}^\infty \frac{x\sin(x)\,dx}{1+x^2}$

Let $f(z) = z/(1+z^2)$ and $g(z) = ze^{iz}/(1+z^2)$. Thus $\operatorname{Im}(g(x)) = f(x)\sin(x)$ when x is real. The only poles for g are simple poles at $\pm i$. Since $f(z) = o(1)$ as $z \to \infty$ we may apply Jordan's Lemma to conclude that

$$\int_{-\infty}^\infty \frac{x\sin(x)\,dx}{1+x^2} = \operatorname{Im}\left(2\pi i \cdot \operatorname{res}(g, i)\right) = \frac{2\pi}{e}.$$

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3. $\displaystyle\int_{-\pi}^\pi \frac{\sin^2(\theta)\,d\theta}{2-\cos(\theta)}$

If we set $z = e^{i\theta}$ then $\sin(\theta) = \frac{1}{2i}(z - z^{-1})$, $\cos(\theta) = \frac{1}{2}(z + z^{-1})$, and $dz = iz\,d\theta$. Hence

$$\begin{aligned}\int_{-\pi}^\pi \frac{\sin^2(\theta)\,d\theta}{2-\cos(\theta)} &= \int_{\kappa(0,1)} \frac{-\frac{1}{4}(z - z^{-1})^2\,dz}{iz(2 - \frac{1}{2}(z + z^{-1}))} \\ &= \frac{1}{2i}\int_{\kappa(0,1)} \frac{(z^2-1)^2\,dz}{z^2(z^2 - 4z + 1)} \\ &= \pi \cdot \left(\operatorname{res}(f, 0) + \operatorname{res}(f, 2 - \sqrt{3})\right),\end{aligned}$$

where $f(z) = \dfrac{(z - z^{-1})^2}{z^2 - 4z + 1}$. There is a double pole at 0, near which we compute that

$$f(z) = (z^{-2} + O(1))(1 + 4z + O(z^2)) = z^{-2} + 4z^{-1} + O(1).$$

Hence $\operatorname{res}(f, 0) = 4$. There is a simple pole at $2 - \sqrt{3}$ of residue

$$\frac{((2-\sqrt{3}) - (2-\sqrt{3})^{-1})^2}{2\cdot(2-\sqrt{3}) - 4} = -2\sqrt{3}.$$

Hence the integral equals $2\pi \cdot (2 - \sqrt{3})$.