Math 4/5/7880, Spring 2013: Answers to the quizzes

Paul Hewitt

April 26, 2013

Quiz 1

Read each question carefully. Use complete sentences. Above all *be neat!*

---

1. What is the modulus of a complex number?

If z is a complex number then its *modulus* $|z|$ is the distance from 0 to z in the complex plane. If x and y are the real and imaginary parts of z then $|z| = \sqrt{x^2 + y^2}$. (Page 10.)

---

2. What is the Triangle Inequality?

If z and w are complex numbers then $|z + w| \leq |z| + |w|$. (Page 11.)

---

3. What is the principal value of the argument of a complex number?

If $z = r(\cos\theta + i\,\sin\theta)$, where $r > 0$ and θ is real, then θ is called an *argument* of z. We denote any such value by $\arg z$. If θ is the unique such value satisfying $-\pi < \theta \leq \pi$ then we call θ the *principal value* of $\arg z$, and denote it by $\operatorname{Arg} z$. (Page 16.)

---

4. What is Euler's Formula?

If θ is real then $e^{i\theta} = \cos\theta + i\,\sin\theta$. (Page 17.)

---

5. What is de Moivre's Formula?

If θ is real and n is an integer then $(\cos\theta + i\,\sin\theta)^n = \cos n\theta + i\,\sin n\theta$. (Page 20.)

Quiz 2

Read each question carefully. Use complete sentences. Above all *be neat!*

---

1. What is a deleted neighborhood of a point in the complex plane?

For any positive value δ the set

$$\{z \mid 0 < |z - z_0| < \delta\}$$

is called a deleted neighborhood of z_0. (Page 31.)

---

2. Suppose that f is a function of a complex variable z, defined in a deleted neighborhood of z_0. What does it mean when we write

$$\lim_{z \to z_0} f(z) = w_0.$$

The equation means that for any positive value ϵ there is a positive value δ such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

(Page 45.)

---

3. What does it mean to say that a function f is continuous at a point z_0?

This means that

$$\lim_{z \to z_0} f(z) = f(z_0).$$

In other words, three things must be true: (1) f has a limit at z_0; (2) f is defined at z_0; (3) the limit at z_0 equals the value of the function there. (Page 53.)

---

4. What does it mean to say that a function f is differentiable at a complex number z_0?

This means that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The value of the limit is usually denoted $f'(z_0)$, and is called the derivative of f at z_0. (Page 56.)

---

5. What is the Riemann sphere?

The Riemann sphere is a sphere whose points are in one-to-one correspondence with the *extended complex plane*, obtained by adjoining the point at infinity to the ordinary complex plane. The correspondence is achieved by means of *stereographic projection.* (Page 51.)

Quiz 3

Read each question carefully. Use complete sentences. Above all *be neat!*

---

1. Suppose that f is a function of a complex variable z, defined in a deleted neighborhood of z_0. What does it mean when we write

$$\lim_{z \to z_0} f(z) = w_0.$$

The equation means that for any positive value ϵ there is a positive value δ such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

(Page 45.)

---

2. What does it mean to say that a function f is continuous at a point z_0?

This means that

$$\lim_{z \to z_0} f(z) = f(z_0).$$

In other words, three things must be true: (1) f has a limit at z_0; (2) f is defined at z_0; (3) the limit at z_0 equals the value of the function there. (Page 53.)

---

3. What does it mean to say that a function f is differentiable at a complex number z_0?

This means that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The value of the limit is usually denoted $f'(z_0)$, and is called the derivative of f at z_0. (page 56.)

---

4. What are the Cauchy-Riemann equations?

If u and v are the real and imaginary parts of a complex function f then the Cauchy-Riemann equations are

$$u_x = v_y, \; u_y = -v_x.$$

Here subscripts indicate partial derivatives. (Page 65.)

---

5. What is an analytic function?

If f is differentiable at every point in some open set then we say that it is analytic on that set. (Page 73.)

Quiz 4

Read each question carefully. Use complete sentences. Above all *be neat!*

---

1. What is an analytic function?

If f is differentiable at every point in some open set then we say that it is analytic on that set. (Page 73.)

---

2. What are the Cauchy-Riemann equations in rectangular coordinates?

If u and v are the real and imaginary parts of a complex function f then the Cauchy-Riemann equations in rectangular coordinates are

$$u_x = v_y, \; u_y = -v_x.$$

(Page 65.)

---

3. What are the Cauchy-Riemann equations in polar coordinates?

If u and v are the real and imaginary parts of a complex function f then the Cauchy-Riemann equations in polar coordinates are

$$r\, u_r = v_\theta, \; r\, v_r = -u_\theta.$$

(Page 69.)

---

4. Compute u_x, u_y, v_x, and v_y, where u and v are the real and imaginary parts of e^z.

Since $u = e^x \cos y$ and $v = e^x \sin y$ we find that

$$u_x = v_y = e^x \cos y, \; u_y = -v_x = e^x \sin y.$$

(Page 42.)

---

5. Compute u_r, u_θ, v_r, and v_θ, where u and v are the real and imaginary parts of $\log z$.

Since $u = \ln r$ and $v = \theta$ we find that

$$u_r = 1/r, \; v_\theta = 1, \; v_r = u_\theta = 0.$$

(Page 93.)

Quiz 5

Read each question carefully. Use complete sentences. Above all *be neat!*

---

1. What are Euler's Formula and de Moivre's Formula?

If θ is real then Euler's Formula says that

$$e^{i\theta} = \cos\theta + i\,\sin\theta.$$

If n is an integer then de Moivre's Formula says that

$$(\cos\theta + i\,\sin\theta)^n = \cos n\theta + i\,\sin n\theta.$$

(Pages 17 and 20.)

---

2. What does it mean to say that a function f is continuous at a point z_0?

This means that

$$\lim_{z\to z_0} f(z) = f(z_0).$$

In other words, three things must be true: (1) f has a limit at z_0; (2) f is defined at z_0; (3) the limit at z_0 equals the value of the function there. (Page 53.)

---

3. What are the Cauchy-Riemann equations, in both rectangular and polar coordinates?

If u and v are the real and imaginary parts of a complex function f then the Cauchy-Riemann equations in rectangular coordinates are

$$u_x = v_y,\ u_y = -v_x.$$

The Cauchy-Riemann equations in polar coordinates are

$$r\,u_r = v_\theta,\ r\,v_r = -u_\theta.$$

(Pages 65 and 69.)

---

4. Compute u_x, u_y, v_x, and v_y, where u and v are the real and imaginary parts of $\sin z$.

If x and y are the real and imaginary parts of z then by the addition formula for sine and the identities $\cos(iy) = \cosh y$ and $\sin(iy) = i\,\sinh y$ we find that

$$\sin(x + iy) = \sin x\,\cosh y + i\,\cos x\,\sinh y.$$

That is, $u = \sin x\,\cosh y$ and $v = \cos x\,\sinh y$. Hence

$$u_x = v_y = \cos x\,\cosh y,\ u_y = -v_x = \sin x\,\sinh y.$$

(Page 106.)

---

5. Compute $|\sin z|^2$ and simplify.

From the above we find that

$$|\sin z|^2 = \sin^2 x\,\cosh^2 y + \cos^2 x\,\sinh^2 y = \sin^2 x\,\cosh^2 y + (1 - \sin^2 x)\,\sinh^2 y = \sin^2 x + \sinh^2 y.$$

Here we have used the identity $\cosh^2 y - \sinh^2 y = 1$. (Page 106.)

Quiz 6

Read each question carefully. Use complete sentences. Above all *be neat!*

---

1. True or false: If t is a real variable then

$$\frac{d}{dt}e^{it} = i\,e^{it}.$$

Explain!

This is *true*. By definition, if u and v are real-valued functions of the real variable t then

$$\frac{d}{dt}(u(t) + i\,v(t)) = \frac{du}{dt} + i\,\frac{dv}{dt}.$$

If we apply this to the exponential then we obtain

$$\frac{d}{dt}e^{it} = \frac{d}{dt}(\cos t + i\,\sin t) = -\sin t + i\,\cos t = i\,e^{it}.$$

(Pages 117–118.)

---

2. True or false: If $w(t)$ is a complex-valued function of a real variable t, defined and continuous on an interval $[a,b]$ then

$$\operatorname{Re}\int_a^b w(t)\,dt = \int_a^b \operatorname{Re}(w(t))\,dt.$$

Explain!

This is *true*, by definition of the complex definite integral. (Page 119.)

---

3. What is an *arc* in the complex plane?

An arc is a pair $(x(t), y(t))$ of real-valued continuous functions defined on some interval $[a,b]$. Thus it defines a parametric representation of a curve of some sort in the complex plane. (Page 122.)

---

4. What is a *simple* arc in the complex plane?

A simple arc $z(t)$ is one that does not "cross itself". That is, if $z(t_1) = z(t_2)$ then $t_1 = t_2$. One must take care when employing the term *simple closed* arc, for in that case one allows that the beginning and ending points are the same, but no other point is repeated. (Page 122.)

---

5. What is a *smooth* arc in the complex plane?

An arc $z(t)$ is *smooth* in case it has a nonvanishing derivative dz/dt are every point $a < t < b$. For technical reasons it is useful not to worry about the existence of the derivative at the endpoints. (Pages 124–125.)

Quiz 7

Read each question carefully. Use complete sentences. Above all *be neat!*

---

1. How is the contour integral $\int_C f(z)\,dz$ defined? Be precise!

If C is given parametrically by the piece-wise smooth function $z(t)$, for $t_1 \le t \le t_2$, then

$$\int_C f(z)\,dz = \int_{t_1}^{t_2} f(z(t))\,\frac{dz}{dt}\,dt.$$

(Page 127.)

---

2. True or false: if f is an analytic function in some domain D and if C is a closed contour in D then $\int_C f(z)\,dz = 0$. Explain!

This is false. For example

$$\int_{|z|=1} \frac{dz}{z} = \int_0^{2\pi} \frac{i\,e^{i\theta}\,d\theta}{e^{i\theta}} = 2\pi i \neq 0.$$

(Compare with Example 3, pages 143–145.)

---

3. True or false: if f is an analytic function in some domain D and if $\int_C f(z)\,dz = 0$ for every closed contour C in D then f has an antiderivative in D. Explain!

This is true. It is part of the theorem characterizing the existence of an antiderivative. (Page 142.)

---

4. True or false: if f is an analytic function in some domain D and if f has an antiderivative in D then for any contour C in D the integral $\int_C f(z)\,dz$ depends only on the endpoints of C. Explain!

This is true. It is part of the theorem characterizing the existence of an antiderivative. (Page 142.)

---

5. State the Cauchy-Goursat Theorem.

If f is analytic at all points inside and on the simple closed contour C then

$$\int_C f(z)\,dz = 0.$$

(Page 151.)

Quiz 8

Read each question carefully. Use complete sentences. Above all *be neat!*

---

1. What does it mean to say that the series $\sum_{n=0}^{\infty} z_n$ converges?

We say that $\sum_{n=0}^{\infty} z_n$ converges to S when $S = \lim_{N\to\infty} \sum_{n=0}^{N} z_n$. (Page 184.)

---

2. True or false: if $\sum_{n=0}^{\infty} z_n$ converges then $\lim_{n\to\infty} z_n = 0$. Explain!

This is true: this is the content of the corollary on page 185. For an alternate proof let $S_N = \sum_{n=0}^{N} z_n$. If $\epsilon > 0$ and we choose M so that $|S_N - S| < \epsilon/2$ when $N > M$ then when $n > M + 1$ we have

$$|z_n| = |S_n - S_{n-1}| \le |S_n - S| + |S_{n-1} - S| < 2\epsilon/2 = \epsilon.$$

---

3. True or false: if $\lim_{n\to\infty} z_n = 0$ then $\sum_{n=0}^{\infty} z_n$ converges. Explain!

This is false. For example the famous harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Indeed,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^n} > 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + \cdots + 2^{n-1} \cdot \frac{1}{2^n} > 1 + \frac{n}{2} \to \infty.$$

---

4. True or false: if $\sum_{n=0}^{\infty} |z_n|$ converges then $\sum_{n=0}^{\infty} z_n$ converges. Explain!

This is true, according to the corollary on page 186. The statement that absolute convergence implies convergence is in fact equivalent to the completeness axiom of the real numbers.

---

5. True or false: if $\sum_{n=0}^{\infty} z_n$ converges then $\sum_{n=0}^{\infty} |z_n|$ converges. Explain!

This is false. For example the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges but the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Quiz 9

Read each question carefully. Use complete sentences. Above all *be neat!*

---

1. What is an isolated singularity of a complex function f? Give an example.

If f is analytic in a deleted neighborhood $\{z \mid 0 < |z - z_0| < R\}$, for some positive R, but is not analytic at z_0, then f is said to have an isolated singularity at z_0. (See page 229.) An important class of examples are the functions $f(z) = 1/z^n$, for $n = 1, 2, 3, \ldots$, which all have an isolated singularity at 0 (but are analytic elsewhere).

---

2. Suppose that f has an isolated singularity at z_0. What is the *definition* of the residue of f at z_0?

By definition of isolated singularity (see above) f is analytic in a deleted neighborhood of z_0 — say $\{z \mid 0 < |z - z_0| < R\}$, for some positive R. By Laurent's Theorem f can be expanded in a power series using positive and negative powers, valid in this domain:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

The *residue* of f at z_0 is defined on page 231 to be a_{-1}. (The book uses the notation b_n for a_{-n}.)

---

3. Suppose that f has an isolated singularity at z_0. What is the *integral formula* for the n-th coefficient of the Laurent expansion of f in a deleted neighborhood of z_0?

As above, suppose that f is analytic in the deleted neighborhood $\{z \mid 0 < |z - z_0| < R\}$. If C is a simple closed contour in R, oriented counterclockwise, with z_0 in its interior then

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}.$$

(See page 219 or page 231.)

---

4. State Cauchy's Residue Theorem.

If f is analytic inside and on a simple closed contour C, oriented counterclockwise, except for a finite number of isolated singularities $z_1, \ldots, z_k$ in the interior of C, then

$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z).$$

(See page 235.)

---

5. True or false: If f is continuous in a domain D, if C is a simple closed contour in D, and if $\int_C f(z)\,dz = 0$ then f is analytic inside and on C. Explain!

This is *false*. For example take $f(z) = 1/z^2$, D the set of nonzero complex numbers, and C the unit circle. In this case the integral is 0 since that is the residue of f at the isolated singularity 0.

However it is true that if the integral is 0 *every* such C in a domain D then f is analytic in D. This is the content of Morera's Theorem.

Quiz 10

Read each question carefully. Use complete sentences. Above all *be neat!*

---

1. State Cauchy's Residue Theorem.

If f is analytic inside and on a simple closed contour C, oriented counterclockwise, except for a finite number of isolated singularities $z_1, \ldots, z_k$ in the interior of C, then

$$\int_C f(z)\, dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z).$$

(See page 235.)

---

2. Suppose that f has an isolated singularity at z_0. What is the integral formula for the n-th coefficient of the Laurent expansion of f in a deleted neighborhood of z_0?

As above, suppose that f is analytic in the deleted neighborhood $\{z \mid 0 < |z - z_0| < R\}$. If C is a simple closed contour in R, oriented counterclockwise, with z_0 in its interior then

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^{n+1}}.$$

(See page 219 or page 231.)

---

3. What are the three types of isolated singularity?

An isolated singularity is either removable, a pole, or essential. (See pages 241–242.)

---

4. What characterizes an essential singularity?

There are at least two ways to characterize an essential singularity z_0.

- The function f is unbounded in every deleted neighborhood of z_0 but $|f(z)|$ does *not* tend to infinity (or any other limit) as $z \to z_0$. (See pages 258–259.)
- The principal part of the Laurent expansion of f is a deleted neighborhood of z_0 has infinitely many terms. (See pages 241–242.)

---

5. What characterizes a pole?

There are at least two ways to characterize a pole z_0.

- $|f(z)| \to \infty$ as $z \to z_0$. (See pages 258–259.)
- The principal part of the Laurent expansion of f is a deleted neighborhood of z_0 is nonzero but has only finitely many terms. (See pages 241–242.)

Quiz 11

Read each question carefully. Use complete sentences. Above all *be neat!*

---

1. Suppose f is a continuous function defined on the entire real line. What is the definition of the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$?

By definition

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to+\infty}\int_0^b f(x)\,dx$$

provided both limits exist. (See page 261.)

---

2. Suppose f is a continuous function defined on the entire real line. What is the definition of the Cauchy Principal Value $\mathrm{P.V.}\int_{-\infty}^{\infty} f(x)\,dx$?

By definition

$$\mathrm{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to+\infty}\int_{-a}^{a} f(x)\,dx$$

provided the limit exists. (See page 262.)

---

3. True or false: if f is a continuous function defined on the entire real line and if $\int_{-\infty}^{\infty} f(x)\,dx$ exists then $\mathrm{P.V.}\int_{-\infty}^{\infty} f(x)\,dx$ exists. Explain!

This is true: if both limits in the definition of improper integral exist then by the additive properties of integrals and limits we find that

$$\lim_{a\to+\infty}\int_{-a}^{a} f(x)\,dx = \lim_{a\to+\infty}\left(\int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx\right) = \lim_{a\to+\infty}\int_{-a}^{0} f(x)\,dx + \lim_{a\to+\infty}\int_0^a f(x)\,dx$$

(See page 262.)

---

4. True or false: if f is a continuous function defined on the entire real line and if $\mathrm{P.V.}\int_{-\infty}^{\infty} f(x)\,dx$ exists then $\int_{-\infty}^{\infty} f(x)\,dx$ exists. Explain!

This is false: for example if $f(x) = x$ then neither of the limits in the definition of improper integral exist but each symmetric integral $\int_{-a}^{a} f(x)\,dx = 0$, and so the Cauchy Principal Value is 0. (See page 262.)

However, if f were an *even* function then each of the integrals in the definition of the improper integral would equal one half of the Cauchy Principal Value. Hence with this extra hypothesis the existence of latter implies the existence of the former, with equality between the values. (See pages 262–263.)

---

5. What is Jordan's inequality?

Jordan's inequality says simply that if $R > 0$ then

$$\int_0^{\pi} e^{-R\sin\theta}\,d\theta \le \pi/R.$$

This is the key inequality used in the proof of Jordan's Lemma. (See pages 272–273.)

Quiz 12

Read each question carefully. Use complete sentences. Above all *be neat!*

---

1. What is Jordan's inequality?

Jordan's inequality says simply that if $R > 0$ then

$$\int_0^\pi e^{-R\sin\theta}\,d\theta \le \pi/R.$$

This is the key inequality used in the proof of Jordan's Lemma. (See pages 272–273.)

---

2. What is Jordan's Lemma?

Let Γ_R denote the upper half of the circle where $|z| = R$. Jordan's Lemma makes the following technical assumptions about the complex function $f(z)$:

- There is a positive constant R_0 such that f is analytic in the upper half plane outside Γ_{R_0}.
- There are positive values M_R tending to 0 as $R \to \infty$ such that $|f(z)| < M_R$ for all $z \in \Gamma_R$.

Under these hypotheses Jordan's Lemma asserts that for any positive a we have that

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)e^{iaz}\,dz = 0$$

(See page 272.)

---

3. What is a meromorphic function?

A complex function is meromorphic in a domain D if it is analytic except possibly for poles in D. (See page 291.)

---

4. What is the Argument Principle?

If f is analytic analytic and nonzero along a simple closed contour C, oriented positively, and if f is meromorphic in the interior of C, then

$$\text{winding number of } f(C) \text{ around } 0 = Z - P$$

where

$$\begin{aligned} Z &= \text{number of zeroes of } f \text{ inside } C \text{ (counting multiplicity)} \\ P &= \text{number of poles of } f \text{ inside } C \text{ (counting multiplicity)} \end{aligned}$$

(See page 292.)

---

5. What is Rouché's Theorem?

If f and g are analytic inside and on a simple closed contour C and if $|f(z)| > |g(z)|$ for every $z \in C$ then f and $f + g$ have the same number of zeroes inside C, counting multiplicity. (See page 294.)

Quiz 13

Read each question carefully. Use complete sentences. Above all *be neat!*

---

1. What is a meromorphic function?

A complex function is meromorphic in a domain D if it is analytic except possibly for poles in D. (See page 291.)

---

2. What is the Argument Principle?

If f is analytic analytic and nonzero along a simple closed contour C, oriented positively, and if f is meromorphic in the interior of C, then

$$\text{winding number of } f(C) \text{ around } 0 = Z - P$$

where

$$Z = \text{number of zeroes of } f \text{ inside } C \text{ (counting multiplicity)}$$
$$P = \text{number of poles of } f \text{ inside } C \text{ (counting multiplicity)}$$

(See page 292.)

---

3. What is Rouché's Theorem?

If f and g are analytic inside and on a simple closed contour C and if $|f(z)| > |g(z)|$ for every $z \in C$ then f and $f + g$ have the same number of zeroes inside C, counting multiplicity. (See page 294.)

---

4. What is a linear fractional transformation?

A linear fractional transformation is a transformation of the form

$$w = \frac{az + b}{cz + d}, \text{where } ad - bc \neq 0.$$

(See page 319.)

---

5. How do linear fractional transformations transform lines and circles?

Any linear fractional transformation transforms a line into either a line or a circle and a circle into either a line or a circle. (See page 319.)

Quiz 14

Read each question carefully. Use complete sentences. Above all *be neat!*

---

1. What is a linear fractional transformation?

A linear fractional transformation is a transformation of the form

$$w = \frac{az+b}{cz+d}, \text{where } ad - bc \neq 0.$$

(See page 319.)

---

2. How do linear fractional transformations transform lines and circles? *Be precise!*

Any linear fractional transformation transforms a line into either a line or a circle and a circle into either a line or a circle. (See page 319.)

---

3. (4 points) Which of the following are conformal? Explain!

a. Linear fractional transformations.

The LFT $w = (az+b)/(cz+d)$ is conformal at every point because its derivative $w' = (ad-bc)/(cz+d)^2$ is nonzero at each point. (See the next item.)

b. Analytic functions.

An analytic function $w = f(z)$ defines a transformation that is conformal at any noncritical point — that is, at any point where $f'(z) \neq 0$. (See page 357.)

c. Stereographic projection.

Stereographic projection is conformal at every point, a fact that was apparently first proved explicitly by Thomas Harriot (1560–1621). Stereographic projection — or the planisphere, as he would have referred to it — was used extensively by Claudius Ptolemy (approx. 85–165) in his extremely influential astronomical work *Almagest*. This work was based on the geometry and trigonometry of Hipparchus (approx. −190–−120). Although the proof of conformality is very simple and uses only elementary geometry it appears not to have been unknown to the ancient Greeks.Unfortunately Hipparchus' major works are all lost and we may never know for certain.

d. Complex conjugation.

Complex conjugation is *isogonal*, meaning that it preserves the magnitude of angles but reverses their orientation. (See page 357.)

Quiz 15

Read each question carefully. Use complete sentences. Above all *be neat!*

---

1. Suppose f is a continuous function defined on the entire real line. What is the difference between the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ and the Cauchy Principal Value P. V. $\int_{-\infty}^{\infty} f(x)\,dx$? *Be precise!*

By definition the improper integral

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to+\infty}\int_0^b f(x)\,dx$$

provided both limits exist. (See page 261.) By contrast, the Cauchy Principal Value is by definition

$$\text{P. V.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to+\infty}\int_{-a}^{a} f(x)\,dx$$

provided the limit exists. (See page 262.) If the improper integral exists then the Cauchy Principal Value exits, and the two have the same value. (See page 262.) The converse is not true without extra hypotheses — for example if f is even or positive. (See pages 262–263.)

---

2. What is Jordan's Lemma?

Let Γ_R denote the upper half of the circle where $|z| = R$. Jordan's Lemma makes the following technical assumptions about the complex function $f(z)$:

- There is a positive constant R_0 such that f is analytic in the upper half plane outside Γ_{R_0}.
- There are positive values M_R tending to 0 as $R \to \infty$ such that $|f(z)| < M_R$ for all $z \in \Gamma_R$.

Under these hypotheses Jordan's Lemma asserts that for any positive a we have that

$$\lim_{R\to\infty}\int_{\Gamma_R} f(z)e^{iaz}\,dz = 0$$

(See page 272.)

---

3. What is a meromorphic function?

A complex function is meromorphic in a domain D if it is analytic except possibly for poles in D. (See page 291.)

---

4. What is the Argument Principle?

If f is analytic analytic and nonzero along a simple closed contour C, oriented positively, and if f is meromorphic in the interior of C, then

$$\text{winding number of } f(C) \text{ around } 0 = Z - P$$

where

$$Z = \text{number of zeroes of } f \text{ inside } C \text{ (counting multiplicity)}$$
$$P = \text{number of poles of } f \text{ inside } C \text{ (counting multiplicity)}$$

(See page 292.)

---

5. What is a linear fractional transformation? How do linear fractional transformations transform lines and circles? *Be precise!*

A linear fractional transformation is a transformation of the form

$$w = \frac{az+b}{cz+d}, \text{ where } ad - bc \neq 0.$$

Any linear fractional transformation transforms a line into either a line or a circle and a circle into either a line or a circle. (See page 319.)