Algebraic groups and Lie algebras

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If K is a field then a *Lie algebra over* K is a K-vector space L together with a bilinear "product" $[\,,\,]\colon L \times L \to L$ which satisfies two simple axioms:

- Skew symmetry: for all $x \in L$, $[x, x] = 0$.
- Jacobi identity: for all $x \in L$, the map $y \mapsto [x, y]$ satisfies the "product rule" for derivations. In other words, for all $x, y, z \in L$, $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$.

These were introduced by Sophus Lie in his attempt to find a "Galois theory" of systems of differential equations. They are in effect the "infinitesimal" symmetries of the system.

Lie's contemporary Camille Jordan was at the time making the first in-depth, systematic investigations of finite groups since Galois, and there was a great deal of mutual influence between Lie and Jordan. In parallel with finite groups, the theory of Lie algebras divided between the solvable — those built of abelian layers — and the simple — those with no nontrivial homomorphic images. The work of Jordan and Lie, in their nominally independent domains, both pointed to the conclusion that the simple objects fall into a small number of infinite families, each rather uniform in behavior, together with a small handful of anomalies (called "exceptional" in Lie's world, "sporadic" in Jordan's).

Soon the work of Killing and others culminated in a complete classification of simple Lie algebras over $\mathbb{C}$: there are indeed four infinite families, labeled A_n, B_n, C_n, and D_n, for $n = 1, 2, 3, \ldots$; and five exceptions, labeled E_6, E_7, E_8, F_4, and G_2. The original classification was not very enlightening, and in fact was rather frustrating, since the unexceptional algebras had lovely internal geoemtry, whereas the exceptions remained geometrically mysterious. Soon Weyl, Cartan, and others began a program to rework the classification in a more conceptual, geometric way. This work produced some of the most beautiful and pervasive tools and concepts in all of mathematics: root systems, Weyl groups, Coxeter complexes, and Tits buildings.

The classification of simple Lie algebras made clear a strong connection with algebraic geometry. A *linear algebraic group over* K is a subgroup of the group $\mathrm{GL}_n(K)$ of $n \times n$ matrices over over K which is also an algebraic manifold — that is, it is defined by a system of polynomial equations. The geometric approach to simple Lie algebras leads to a classification of simple linear algebraic groups. In fact, the simple Lie algebras can be recovered from the algebraic groups as the algebra of left-invariant vector fields. The whole of representation theory exploits this relationship in a very fundamental way.

This summer I will offer a course on algebraic groups and Lie algebras. This will be a rapid introduction to the theory of simple linear algebraic groups, their Lie algebras, and their rational representations. In the first part of the semester I will introduce just enough algebraic geometry and commutative algebra to establish the elementary theory of algebraic groups. The middle part will be devoted to Lie algebras and their connection to algebraic groups. In the remaining time I will introduce some of the basic tools of representation theory, including root systems and so forth.

I will not require a book for this course, but for much of the semester I will follow Tonny Springer's book [1]. I will also draw some material from the books of James Humphreys [2] and [3].

References

[1] TA Springer, *Linear Algebraic Groups*. Birkhäuser, 1980.

[2] JE Humphreys, *Linear Algebraic Groups*. Springer: GTM no 21, 1981.

[3] JE Humphreys, *Introduction to Lie Algebras and Representation Theory*. Springer: GTM no 9, 1972.