Graphs and Networks

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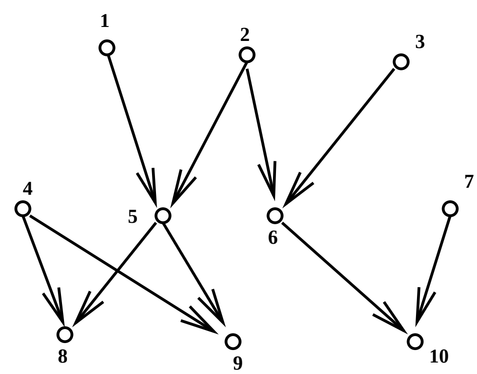
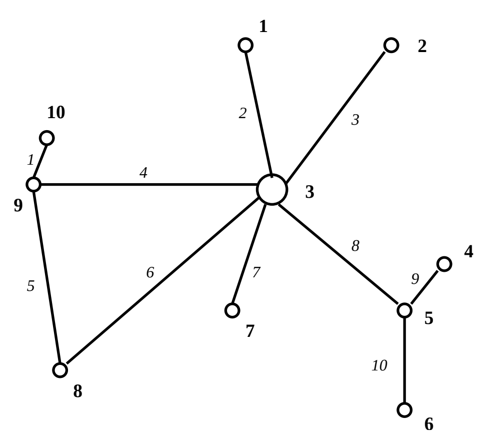
Informally, a *graph* or *network* is a way to represent a binary relationship. Formally, a graph is a set V of *vertices* (also called *nodes*) and a subset E of $V \times V$ called *edges*. Graphs are used in a wide range of situations. For example, geneticists might use a graph to represent a population, with an edge between individuals v_1 and v_2 representing the relationship "v_1 is a parent of v_2". Since this is an asymmetric relationship we would picture this graph with the edges drawn as directed arrows:



Graphs are used by airlines to represent cities that are served by direct flights. In this case the relationship is typically symmetric, and so we picture this graph with undirected edges:



And there are a myriad more applications, from telecommunications to economics to particle physics to sociology.

When we use a computer to work with large graphs we need a way to represent the graph in a way that the computer can store and manipulate. One standard way to do this is with matrices. In fact there are (at least!) two standard matrix representations of a graph, the *adjacency matrix* and the *incidence matrix.* In the adjacency matrix A, the rows and columns are indexed by the vertices, and $A_{ij} = 1$ if vertex i is joined to vertex j, and $A_{ij} = 0$ otherwise. Here is the adjacency matrix for our genealogical example:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

When we have an undirected graph we sometimes use the adjacency matrix and sometimes use the incidence matrix. In the incidence matrix B the rows are again indexed by the vertices but the columns are indexed by the edges, and $B_{ij} = 1$ if vertex i is on edge j, and $B_{ij} = 0$ otherwise.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

One of the most important notions in a graph is a *path*, which is a sequence of vertices $v_0, v_1, \ldots, v_n$ such that v_i is joined to v_{i+1}. A path of length n has $n+1$ vertices. In the airline example, $3 \sim 1 \sim 3 \sim 8 \sim 9$ is a path of length 4. In the genealogical example $5 \to 2 \to 6 \to 10$ is not a path, because we do not have the connection $5 \to 2$.

If Γ is an undirected graph, and v_1 and v_2 are vertices, then the distance between v_1 and v_2 is defined to be the shortest length among all paths joining v_1 and v_2. If there is no path between the vertices then we might say that the distance between them is infinite. The *diameter* of a graph is the largest distance among all pairs of vertices. Thus, a graph is connected if and only if its diameter is finite. For our airline graph, the diameter is 4.

The *valency* of a vertex in an undirected graph is the number of edges joined to it. This is also sometimes called the *degree* of a vertex. For a directed graph we have two valencies at each vertex: the out-valency and the in-valency, or out-degree and in-degree. The *neighborhood* of a vertex v in an undirected graph is the part of the graph containing all vertices which are joined to v. Again, we would have to make adjustments to this definition for a directed graph.

Problems

Note: if there is sufficient interest in this application, then I will post more problems. Let's start with a pretty one.

1. Let A and B be the adjacency and incidence matrices of a graph Γ.

   - Interpret BB^*.
   - Interpret A^k, where k is a positive integer.
   - Discuss the eigenvalues and eigenvectors of A.
   - Now examine ***knight's tours***, the movements of a knight on a chessboard. The vertices of Γ are the squares of an $N \times N$ chessboard. We join vertices v and w when a knight placed at square v can reach square w in one move. Study the diameter of this graph, as a function of N.

2. Suppose you are at a large party where the host tells you that any two guests at the party will find a unique guest who is a common friend. The host is boring so while she is talking you spend your time trying to prove that there must be a unique guest who is friends with everyone...

   - Form a graph Γ where the vertices are the guests and two nodes are joined when they were friends prior to the party. Show that if v and w are not friends of each other then they must have the same number of friends at the party.
   - Let Δ be the subgraph consisting of all nodes with minimum valency. Let Θ consist of all of the other nodes. Show that every node in Δ is joined to every node in Θ.
   - Show that if Θ is nonempty then it consists of a unique node.

   This would finish the proof if you could somehow prove that $\Theta \neq \emptyset$. Since the host is still yakking on and on so you start to think abou linear algebra...

   - Let J denote the $N \times N$ matrix all of whose entries equal 1. What is AJ? JA?
   - Derive a simple expression for A^2.

3. This problem is inspired by the previous one, but is independent of it.

   - Let J denote the $N \times N$ matrix all of whose entries equal 1. Show that J is a scalar multiple of an orthogonal projector. What is its kernel? What is its range?
   - Suppose A is a matrix that commutes with J. Show that A maps $\ker(J)$ to itself and $\text{range}(J)$ to itself.
   - Let A be the $N \times N$ adjacency matrix of an directed graph Γ. Under what conditions does A commute with J?
   - Now let A be the adjacency matrix of an undirected graph. Suppose that $A^2 = tI + uJ$. Interpret t and u.
   - Under the same hypotheses as above, describe A as completely as possible. Are there any implications for N, t, and u?