Fourier Theory: a Model for Sound

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The ear detects sound by responding to the fluctuations of the air pressure in the ear canal. On the ear drum are "hairs" of various length, each of which respond to sound of a particular frequency and its overtones. As the hair vibrates in consonance with the air, the strength of the sound at that frequency is transmitted to the brain by the neuro-electrical impulse generated by the hair. The ear can therefore respond to frequencies only in a limited range.

Mathematically, the ear takes a fluctuating air pressure $f(t)$ and projects it onto the space of functions which can be broken down into component frequencies (and their overtones) in a given range:

$$f(t) \approx a_0 + a_1 \sin(\omega(t - \delta_1)) + \cdots + a_N \sin(N\omega(t - \delta_N)). \tag{1}$$

The most effective projection, the one that will capture the most information, is the one which minimizes the total energy of the error.

Since the energy of a function r is proportional to the square of its integral, it is reasonable to introduce an inner product on the space of continuous functions:

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt.$$

Once we have an inner product we can talk about the length (2-norm) of a function and of the angle between two functions. This might be hard to picture, because the space of continuous functions is infinite dimensional, but this inner product has all of the algebraic properties of vector inner product a^*b, and hence we can apply the geometric intuition of vectors in 2- and 3-space to the study of continuous functions.

In particular, we can use all of the formulas we developed for orthogonal projection, including the Gram-Schmidt algorithm, when we approximate a function by a series such as in equation (1).

Problems

If there is sufficient interest in this application, then I will post more problems.

1. Let $\omega = (2\pi)^{-1}$.

   Show that $\sin(n\omega(t - \delta))$ is a linear combination of $\sin(n\omega t)$ and $\cos(n\omega t)$. Conversely, show that every linear combination of $\sin(n\omega t)$ and $\cos(n\omega t)$ can be written in the form $a \sin(n\omega(t - \delta))$.

- Show that the collection

$$1, \cos(\omega t), \ldots, \cos(N\omega t), \sin(\omega t), \ldots, \sin(N\omega t)$$

is an orthogonal set. Conclude that these functions form a basis for their span, which we will denote T_N.

- Compute the 2-norms of the functions above. Use this to give an explicit formula for the orthogonal projection of a function $f(t)$ onto T_N.

- Rewrite the orthogonal projection in the form

$$f(t) \approx a_0 + a_1 \sin(\omega(t - \delta_1)) + \cdots + a_N \sin(N\omega(t - \delta_N)).$$

- (Extra credit.) Obtain a plot of the air pressure measurements in a small cylindrical tube for the sound produced by a voice or musical instrument. Use your formula for vector projection to compute the strength at all component frequencies up to 20000 cps.

2. In this problem we apply the same ideas to another problem: approximating functions by polynomials. When we naively approximate a function by a polynomial of high degree — for example, using Taylor's theorem — the errors tend to accumulate at the ends of the interval, and the resulting polynomial becomes very large there. This is unsuitable for most applications. So, we introduce a new inner product:

$$\langle f, g \rangle = \int_{-1}^{1} \frac{f(t)g(t)\,dt}{\sqrt{1-t^2}}$$

As always, we define the 2-norm using the inner product: $\|f\|_2^2 = \langle f, f \rangle$. Since $1/\sqrt{1-t^2}$ becomes infinite as $t \to \pm 1$, deviations between functions near these endpoints will be emphasized when determining the norm of their difference.

- Apply the Gram-Schmidt algorithm to the collection $\{1, t, t^2, \ldots, t^N\}$. In fact it is a bit easier to skip the normalization step, and simply to produce polynomials that are mutually orthogonal, but not of norm 1. Label the resulting polynomials $\mathrm{Ch}_0, \mathrm{Ch}_1, \ldots, \mathrm{Ch}_N$. These are called the *Chebyshev polynomials*. (Compute the first few, up to at least degree 6.)

- Show that when $n > 0$ we have the recurrence

$$\mathrm{Ch}_{n+2}(t) = 2t\,\mathrm{Ch}_{n+1}(t) - \mathrm{Ch}_n(t).$$

- Let $f(t) = \sin(\pi t)/(\pi t)$. Find the orthogonal projection of $f(t)$ onto the space of polynomials of degree at most 6. Compare the graph of the resulting polynomial with that of f and with that of the Taylor polynomial of degree 6.

- (Extra credit.) Show that $\cos(nt) = \mathrm{Ch}_n(\cos(t))$.