THE ROOTS OF OUR PLACE-VALUE SYSTEM

PAUL HEWITT, UNIV OF TOLEDO

It took a long time for the decimal numeration system to become the world standard. However, the concept of a positional system has roots deep and wide.

1. Multiplication and division in ancient Egypt

Ancient Egyptians grouped in tens. Their hieratic system — the one used for everyday computation with ink on papyrus, not the one carved into stone — had separate symbols for the nine numbers less than ten; and different symbols for the nine multiples of ten less than one hundred; and different one yet for the nine multiples of one hundred less than one thousand; and so forth. Thus, they could represent any number less than one thousand with (at most) three symbols, somewhat akin to the way we do. They had no symbol for zero, except as a marker on a ruler for the origin of measurement. They typically put their number symbols in the same order as we do: hundreds, then tens, then units. It never occurred to them by use of this fixed order they would only need ten symbols to represent any whole number.

However, their algorithms for multiplication and division reveal a strong link to a positional system, although not in base ten! In effect, their arithmetic system used the principles of the same binary arithmetic that is used in modern computers. They simply didn’t write it out using a sequence of zeroes and ones. This method is sometimes called the “peasant” algorithm, because it remained in widespread use among peasant populations in Eastern Europe well into the 20th century. It is quite efficient, and easy to use. Let’s look at some examples.

We will not write numbers using the hieratic numerals; the point here is to glimpse the algorithm, not the symbols. The simplicity of the algorithm will be seen by using familiar symbols. Suppose we want to multiply 59 × 83.

\[
\begin{array}{c|cc}
83 & 59 \\
- 64 & 2 * 118 \\
--- & 4 236 \\
19 & 8 472 \\
- 16 & 16 * 944 \\
--- & 32 1888 \\
3 & 64 * 3776 \\
- 2 & ---- \\
--- & 4897 \\
1 & \\
\end{array}
\]

Look at the middle column, which we would tabulate once and for all. We repeatedly subtract from 83 these powers of 2, always using the largest possible
subtrahend. We end up with an (implicit) binary representation of 83:

$$83 = 64 + 16 + 2 + 1 = 2^6 + 2^4 + 2^1 + 2^0 = 1010011_{\text{two}}.$$  

The summands are marked with asterisks above. If we multiply both sides by 59 then we get

$$83 \times 59 = 2^6 \times 59 + 2^4 \times 59 + 2^1 \times 59 + 2^0 \times 59.$$  

In other words, we can compute the product merely by repeatedly doubling 59, then adding those results corresponding to the summands of 83 in its binary representation.

The same method works well backwards — that is, when we divide. Suppose instead we are computing 4897 ÷ 59:

\[
\begin{array}{ccc}
4897 & 1 & 59 \\
-3776 & 2 & 118 \\
-521 & 8 & 472 \\
-944 & 16 & 944 \\
-187 & 64 & 3776 \\
-118 & & \\
-59 & & \\
\end{array}
\]

This time, we begin with 4897 and repeatedly subtract values from the third column, again always using the largest possible subtrahend. We conclude that the quotient is the sum of the corresponding powers of 2:

$$4897 \div 59 = 64 + 16 + 2 + 1 = 83.$$  

But what do we do when the quotient is not a whole number? Suppose now that we are computing 800 ÷ 59:

\[
\begin{array}{ccc}
800 & 1 & 59 \\
-472 & 2 & 118 \\
-236 & 4 & 236 \\
328 & 8 & 472 \\
-944 & 16 & 944 \\
92 & 64 & 3776 \\
-59 & & \\
-33 & & \\
\end{array}
\]

So, we might write the result as a “mixed” fraction:

$$800 \div 359 = 8 + 4 + 1 + \frac{33}{59} = 13 \frac{33}{59}.$$  

However, in ancient Egypt they did not have the concept of a common fraction such as 33/59. They expressed fractional amounts using **unit fractions**, which are the reciprocals of whole numbers. In fact, they did not refer to a quantity 1/n as a reciprocal or a fraction of any kind but as the “n-th part”. In other words, you divide a whole into n equal parts and the n-th of these represents 1/n of the whole.
I won’t explain here why the Egyptians might have preferred using unit fractions. In *Crest of the Peacock* Joseph gives some plausible reasons why this was useful in practical applications. Let me simply continue the example above, to show you how you might decompose a common fraction into unit fractions. Since

\[
\frac{33}{59} > \frac{1}{2}
\]

we see that \(\frac{33}{59}\) is \(\frac{1}{2}\) plus a little. Plus how much?

\[
\frac{33}{59} - \frac{1}{2} = 2 \cdot \frac{33}{59} - \frac{59}{2} = \frac{7}{118}.
\]

We can continue this way:

\[
\frac{1}{16} > \frac{7}{118} > \frac{1}{17}
\]

and

\[
\frac{7}{118} - \frac{1}{17} = \frac{1}{2006}.
\]

Thus,

\[
800 \div 359 = 13 + \frac{1}{2} + \frac{1}{16} + \frac{1}{2006}.
\]

Of course, such decompositions were tabulated for easy reference. Unfortunately, we do not know how they constructed these tables. The problem is that there is no unique way to express a fractional quantity as a sum of unit fractions. For example

\[
\frac{2}{17} = \frac{1}{9} + \frac{1}{153} = \frac{1}{12} + \frac{1}{51} + \frac{1}{68}.
\]

It turns out that the Egyptians preferred the second decomposition. Why? There are various guesses, but alas they almost certainly will remain just that. The ancient papyri we have offer too few clues as to how they did this part of the computation.

Nevertheless, we do have copies of their tables, and with these we can see that they were able to use the arithmetic methods above even when confronted with mixed fractions. Essentially all that is needed to extend these methods to the case of fractional amounts is to have at hand a table of the doubles of unit fractions. Since they would not have expressed twice the 5-th part as \(\frac{2}{5}\), the way we do, they would have to look up in their table that

\[
2 \times \frac{1}{5} = \frac{1}{3} + \frac{1}{15}.
\]

You can find more explanation and worked examples at *MacTutor* or in Katz’ *A History of Mathematics*.

2. Ancient Iraq’s Sexagesimal System

The first explicit positional system appeared in ancient Iraq. Its base is 60, not 10. Why 60? We don’t know, but we can make a good guess. There are (at least!) two commonly used ways to count on your fingers:

- Each digit is a unit – hence our double use of the word “digit”. We would expect people who count this to group in fives or tens, and thence to develop a numeration system based on 5 or 10. Of course, the base-10 system is today’s standard, but there remain cultures today which use base-5 numeration.
Use your thumb to count segments on each of the remaining 4 fingers. In this way it is easy to count to 12 on one hand — even today, with the decimal system used almost universally, many people still do this. We would expect people who count this to group in twelves, and thence to develop a numeration system based on 12. We see remnants of this in our division of a foot into 12 inches. An “inch” used to be defined as the length of the middle joint of the index finger.

Some historians conjecture that in ancient Iraq there were some cultures using base 5 or 10 numerals and others using base 12. When they merged — perhaps through conquest, or perhaps through migration and intermingling — it became convenient to use a “common” base. The least common multiple of 10 and 12 is 60.

There is a hint of evidence that this might be the correct explanation. The ancient Iraqis divided day and night into twelve hours each, as we do. They divided the solar year into twelve months of 30 days each (eventually they realized the discrepancy) and divided the circle into an analogous 360 degrees. However, they used numerals for 1 and 10, not 1 and 12.

These symbols were formed by pressing a sharpened reed into soft clay, making wedge-shaped or “cuneiform” marks. They could turn the reed so as to make two different kinds of marks. So they might have written 27 something like this:

\[
\langle \langle \vee \vee \vee \vee \vee \vee \vee
\]

But this does not mean that they grouped in tens. In fact, theirs was a sexagesimal, or base-60 system. They used these cuneiform marks to represent numbers less than 60, and then grouped in sixties. Thus, 267 might look like this:

\[
\vee \vee \langle \langle \langle \langle \vee \vee \vee \vee \vee
\]

In other words, this represents \(4 \times 60 + 27\). This same system was used for both whole numbers and fractions. So, depending on context, the above numeral might also mean \(4 \frac{27}{60}\). In fact, depending on context, the space between the two sexagesimal digits might indicate zero in the sixties place. It might even mean that there is no units digit! Hence the numeral might be interpreted as \(4 \times 60^2 + 27\), or even \(4 \times 60^2 + 27 \times 60\).

This ambiguity never bothered them enough to develop a “sexagesimal point”, the way we use a decimal point, although they occasionally used a complicated symbol of diagonally-oriented wedges to indicate a missing digit. Often they simply used the context to interpret the numeral.

We still use use remnants of this in our use of hours/minutes/seconds in time-keeping and degrees/minutes/seconds in trigonometry. When we write the time as 7 : 13 : 25 we mean that it is \(7 + \frac{13}{60} + \frac{25}{60^2}\) hours after twelve o’clock. For most of human history calendars and time-keeping had their basis in the movements of the “planets” or “wandering stars” (including the sun and moon), and the mathematics used to make this astronomy precise was trigonometry. It is only relatively recently that trigonometry has had any other significant uses.

We close this section by reminding ourselves how to change bases for numerals. We start with whole numbers. Since the essence of a base-n system is to form groups of size \(n\), the last digit merely represents the remainder after division by \(n\). So, to convert the decimal number 13157 we compute that 13157 \div 60\) is 219, with
a remainder of 17. Hence the last sexagesimal digit is 17. To find the rest of the sexagesimal digits we recursively divide with remainder. So, 219 ÷ 60 is 3, with remainder 39. Since 3 < 60 we are done:

\[
13157_{\text{ten}} = 1 \times 10^4 + 3 \times 10^3 + 1 \times 10^2 + 5 \times 10^1 + 7 \times 10^0 \\
= 3 \times 60^2 + 39 \times 60^1 + 17 \times 60^0 \\
= 3,39,17_{\text{sixty}}.
\]

Or, in cuneiform:

\[
\bigvee \bigvee \bigvee \bigvee \bigvee \bigvee \bigvee \bigvee
\]

For fractional amounts we reason similarly, but we end up using multiplication, rather than division. This is because a decimal such as 0.125 represents a number that is slightly bigger than one tenth. Indeed \(10 \times 0.125 = 1\) plus a fractional amount, 0.25. Thus, the algorithm for computing the base-\(n\) digits of a fractional amount is to repeatedly multiply by \(n\) and then subtract off the fractional part.

Let’s convert the decimal 0.125 into sexagesimal, and also into binary. Since \(60 \times 0.125 = 7.5\) we see that the first sexagesimal digit is 7. Since \(60 \times 0.5 = 30\) we see that the next sexagesimal digit is 30. The process ends here, since there is no fractional part at this point. Generally the process will repeat “forever” — meaning as far as we need, for a particular application. In summary:

\[
0.125_{\text{ten}} = 1 \times 10^{-1} + 2 \times 10^{-2} + 5 \times 10^{-3} \\
= 7 \times 60^{-1} + 30 \times 60^{-2} \\
= 0.7,30_{\text{sixty}}.
\]

Or, in cuneiform:

\[
\bigvee \bigvee \bigvee \bigvee \bigvee \bigvee \bigvee
\]

Now on to binary: \(2 \times 0.125 = 0.25\), and so the first binary digit is 0. So is the second, since \(2 \times 0.25 = 0.5\). Finally, \(2 \times 0.5 = 1\), and hence

\[
0.125_{\text{ten}} = 1 \times 10^{-1} + 2 \times 10^{-2} + 5 \times 10^{-3} \\
= 0 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} \\
= 0.001_{\text{two}}.
\]

3. The medieval Chinese counting board

The first use of a decimal positional system was in China. They did not use a positional system to write their numbers. They used a system akin to the Egyptian hieratic system, although with different symbols. However, they invented and ingenious counting board, using rods which they would orient either vertically or horizontally to represent 1 or 5. To facilitate rapid calculation and to avoid some of the ambiguities of the Iraqi system they alternated the orientation of the rods in consecutive digits. Thus,

\[
|-
\]

represents 68, since the “|” represents 5 in the leading digit and 1 in the last digit, while the “—” represents the opposite. In effect, theirs was a base-100 system.

The Chinese also worked with both positive and negative numbers, using black and red rods. We still represent negative accounts “in the red”.
They also discovered the binomial theorem, and used this together with their counting boards to compute the digits of square and cube roots, as well as the roots of more complicated polynomial equations.

4. DECIMALS AND EXPONENTS IN MEDIEVAL INDIA

The first instance we have of numerals in essentially their modern form is on a temple engraving in Cambodia. The ancient Cambodian kingdom at Angkor Wat was founded by Hindu princes. The Hindus used a numerations system akin to that used in ancient Egypt. The symbols for the first nine digits are the precursors to our own modern decimal digits. Some historians speculate that increasing contact with Chinese merchants and their counting boards inspired them to abandon the symbols for ten, twenty, and so forth, and simply use their unit digits in a positional, base-10 system. To this they added an important ingredient that was introduced only twice in history: a symbol for zero, not just to avoid the ambiguities of a positional system, but also to represent a number in its own right. Only the Mayans had this same fundamental advance.

This decimal system spread rapidly across Eurasia, first to the Arab empire, and thence eventually to Europe. The numerals evolved, and the European version eventually became the world standard in the colonial era.

However, the binary system that dates back to ancient Egypt remained useful. In medieval India it was adapted to efficiently compute powers by repeated squaring. Suppose we wish to compute $3^{83}$. We make a table similar to that found on Egyptian papyri, but rather than repeatedly doubling 3 we instead repeatedly square it:

\[
\begin{array}{cccc}
83 & 1 & 3 \\
- 64 & 2 & 9 \\
--- & 4 & 81 \\
19 & 8 & 6561 \\
- 16 & 16 & 43046721 \\
--- & 32 & 185302018851841 \\
3 & 64 & 3433683820292512484657849089281 \\
- 2 & \ldots \ldots \ldots \\
--- & 171792506910670443678820376588540424234035840667 \\
1 & \ldots \ldots \ldots \\
\end{array}
\]

We arrive at the final result by multiplying together the quantities corresponding to the binary digits of 83. In other words

\[
3^{83} = 3^{2^6+2^4+2+1} = 3^{2^6} \times 3^{2^4} \times 3^2 \times 3 = (((((3^2)^2)^2)^2)^2 \times (((3^2)^2)^2))^2 \times 3^2 \times 3
\]

Although this may seem strange to you, note how efficient it is: rather than the 82 multiplications to compute $3 \times 3 \times \cdots \times 3$ we used only 9 multiplications, including the 6 squarings.

This method is the one used today in electronic computers. In fact, almost all arithmetic performed today is much closer to the Egyptian method than to the method we are taught in grade school. That’s because almost all arithmetic today is done not by humans but by electronic computers.
Exercises

(1) Use the ancient Egyptian method to perform the following computations. Explain why the methods work, using the distributive property, as above. (You may want to consult either MacTutor or Katz’ book.)

(a) $113 \div 61$
(b) $25\frac{1}{2} \times 42\frac{3}{5}$
(c) $3\frac{1}{2} \div 1\frac{1}{4}$

(2) Make up symbols for base 7, and write out the addition and multiplication tables in your symbols. Write out two 5-digit base-7 numbers. Multiply them, using your tables. Convert your factors to base 10, then multiply them again. Convert your new result to base 7. Do your two answers agree?

(3) Look up the circumference of the earth to the nearest mile, and express this number in sexagesimal, binary, and base 7. (Use the symbols from the previous problem.) Express the constant $\pi$ in these same bases. Use enough digits so that the error is less than $10^{-20}$ in each representation.

(4) Use the Indian method to compute $7^{22}$ and $13^9$. Explain why the methods work, using the properties of exponents, as above. Count how many multiplications you performed, and compare this to how many you would have to do the naive way.