7 Completely reducible modules

Suppose that $M$ is an $R$-module. We say that $M$ is **decomposable** when $M = K \oplus L$, where $K$ and $L$ are nonzero. (In particular, decomposable modules are nonzero.) We say that $M$ is **indecomposable** when it is nonzero but not decomposable. We say that $M$ is **simple** — or **irreducible** — when it has exactly two submodules, namely $0$ and $M$. (In particular, simple modules are nonzero.) We say that $M$ has **finite length** when it contains a finite filtration

$$M = M_0 > M_1 > \cdots > M_n = 0$$

with simple factors $M_j/M_{j+1}$. Such a filtration is called a **composition series**, and the factors are called the **composition factors**. The famous Jordan-Hölder Theorem (which we will not use) states that any two composition series in a given module $M$ have the same length and the same factors, but in possibly different orders. Although we will not prove the Jordan-Hölder Theorem we will sketch a couple of the important steps.

**Lemma 7.1** (Dedekind’s Modular Law). Suppose that $A$, $B$, and $C$ are $R$-modules. If one of the three is contained in one of the other two then

$$(A + B) \cap C = A \cap C + B \cap C.$$

**Proof.** If either $A > C$ or $B > C$ then both sides of this equation equal $C$. The interesting case is where (say) $C > A$. In this case . . .

**Lemma 7.2** (Schreier-Zassenhaus Refinement Theorem). Any two finite filtrations of a module $M$ have equal-length refinements with the same nonzero factors (counting multiplicity, but possibly in different orders).

**Proof.** Suppose

$$M = K_0 > \cdots > K_m = 0$$

and

$$M = L_0 > \cdots > L_n = 0$$

are two refinements of $M$. For each $i \in [0, m]$ and $j \in [0, n]$ define

$$K_{i,j} = K_i + K_{i-1} \cap L_j$$

and

$$L_{j,i} = L_j + L_{j-1} \cap K_i.$$
By Noether’s isomorphism and the Modular Law we see that

\[
\frac{K_{i,j}}{K_{i,j+1}} = \frac{K_{i+1} + K_i \cap L_j}{K_{i+1} + K_i \cap L_{j+1}} \\
\cong \frac{K_i \cap L_j}{K_i \cap L_j \cap (K_{i+1} + K_i \cap L_{j+1})} \\
= \frac{K_{i+1} \cap L_j + K_i \cap L_{j+1}}{K_i \cap L_j} \\
\cong \frac{L_{j,i}}{L_{j,i+1}}.
\]

Notice that if \( K \) is a field and \( R \) is a \( K \)-algebra then any \( R \)-module which is finite-dimensional over \( K \) has a composition series. A module has a composition series precisely when it is both noetherian and artinian.

**Lemma 7.3.** If \( I \) is a minimal left ideal in \( R \) then either \( I^2 = 0 \) or else \( I = Re \) for some idempotent \( e \).

**Proof.** Suppose \( Ia \neq 0 \) for some \( a \in I \). Minimality implies that \( Ia = I \). Choose \( e \in I \) such that \( ea = a \). Let \( J = \{ b \in I \mid ba = 0 \} \). By minimality again, \( J = 0 \). Since \((e^2 - e)a = e^2a - ea = ea - ea = 0\) we conclude that \( e^2 = e \). Since \( 0 \neq 1e < I \) we must also have that \( I = Ie \).

**Lemma 7.4.** If \( I \) is a left ideal in \( R \) then \( I \) is a direct summand if and only if \( I = Re \) for some idempotent \( e \). In this case, we also have that \( I = Ie \), and that \( I \) is a ring with identity \( e \).

**Proof.** Suppose that \( I = Re \) where \( e^2 = e \). Set \( f = 1 - e \) and \( J = Rf \). Note that \( fe = ef = e - e^2 = 0 \). If \( x \in R \) then \( x = x \cdot 1 = xe + xf \in I + J \). If \( xe = yf \) then \( xe = xe^2 = yfe = 0 \). Hence \( R = I \oplus J \).

Conversely, suppose that \( R = I \oplus J \). Say \( 1 = e + f \), where \( e \in I \) and \( f \in J \). If \( x \in R \) then \( x = xe + xf \). Since \( R = I \oplus J \), \( xe \in I \), and \( xf \in J \), we conclude that if \( x \in I \) then \( xe = x \). In particular, \( e^2 = e \) and \( I = Ie \). Finally, \( Re < I = Ie < Re \), whence \( I = Re \).

A module is **completely reducible** if every submodule is a direct summand.

**Theorem 7.5.** Suppose that \( M \) is an nonzero \( R \)-module. The following are equivalent.

1. \( M \) is a sum of simple modules.
2. \( M \) is a direct sum of simple modules.
3. \( M \) is completely reducible.

We prove this using a couple of lemmas.
Lemma 7.6. Suppose that \( A \) is completely reducible. If \( B \) is a submodule then both \( B \) and \( A/B \) are completely reducible.

**Proof.** If \( C < B \) then \( A = C \oplus D \) for some \( D \). Let \( E = D \cap B \). We have that \( C \cap E < C \cap D = 0 \) and
\[
C + E = C \cap B + D \cap B = (C + D) \cap B = A \cap B = B,
\]
by the Modular Law. Hence \( B = C \oplus E \).

Now if \( A = B \oplus F \) then \( F \cong A/B \), and \( F \) is completely reducible. \( \square \)

Lemma 7.7. Suppose that the \( R \)-module \( A \) is generated by a family of simple submodules. Say
\[
A = \sum_{i \in I} B_i,
\]
where each \( B_i \) is simple. If \( C < A \) then there is a \( J \subset I \) such that
\[
A = C \oplus \bigoplus_{j \in J} B_j.
\]

In particular (taking \( C = 0 \)) we conclude that \( A \) is the direct sum of some subfamily of the \( B_i \).

**Proof of theorem 7.5.** 1 \( \implies \) 2: This follows from lemma above, taking \( C = 0 \).

2 \( \implies \) 3: If \( C < M \) then the lemma above implies that \( C \) is a direct summand.

3 \( \implies \) 1: Let \( S \) be the sum of all the simple modules in \( M \). Suppose that \( S \neq M \): we seek a contradiction. Let \( 0 \neq x \in M \). Choose \( N \) be maximal with respect to the following two properties:

1. \( x \notin N \).
2. \( N > S \).

(This is a straightforward application of Zorn’s Lemma.) Thus, \( N \) is a maximal submodule of \( Rx + N \), and \( (Rx + N)/N \) is simple. Since \( M \) is completely reducible, so is \( Rx + N \). Thus, \( Rx + N = K \oplus N \), where \( K \) is simple. This contradicts the choices above. Hence \( S = M \), as claimed. \( \square \)