Colon ideals and primary decomposition

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Let \( R \) be a (unitary!) commutative ring. If \( I \) and \( J \) are ideals in \( R \) we let

\[
[I : J] = \{ x \in R \mid xJ \subset I \}.
\]

(1)

So, \([I : J]\) is the largest ideal \( K \) such that \( JK \subset I \). Better notation might be \( I/J \), but that notation is already taken.

It is straightforward to check that \([I : J]\) is an ideal. Note that \( I \subset [I : J] \).

So, if \( F \) is a field and \( R = F[x_1, \ldots, x_n] \) then \( \mathbb{Z}([I : J]) \subset \mathbb{Z}(I) \). Moreover, \( \mathbb{Z}(I) - \mathbb{Z}(J) \subset \mathbb{Z}([I : J]) \). Indeed, let \( p \in \mathbb{Z}(I) - \mathbb{Z}(J) \). Choose \( f \in J \) such that \( f(p) \neq 0 \). If \( g \in [I : J] \) then \( fg \in I \), whence \( f(p)g(p) = 0 \). We conclude that \( g(p) = 0 \). In fact, if \( F \) is algebraically closed then \( \mathbb{Z}([I : J]) \) is the closure of \( \mathbb{Z}(I) - \mathbb{Z}(J) \). (See exercise 54 in section 15.2.)

If \( a \in R \) we write \([I : a]\) for \([I : aR]\).

Note that \( \bigcap_i I_i : J = \bigcap_i [I_i : J] \). Indeed, \( xJ \subset \bigcap_i I_i \) if and only if \( xJ \subset I_i \) for all \( i \).

Suppose that \( R \) is noetherian. To establish that every ideal in \( R \) has a minimal (or irredundant) primary decomposition we use the acc to first prove that every ideal has a decomposition into irreducibles, and then we use colon ideals to prove that an irreducible ideal is primary. We have already proven that for any prime \( P \) and any finite collection \( Q_1, \ldots, Q_n \) of \( P \)-primary ideals we have that \( \bigcap_i Q_i \) is also \( P \)-primary. So, we obtain a minimal primary decomposition from a decomposition into irreducibles by first replacing all of the irreducibles with the same radical by their intersection, and then eliminating any redundant terms remaining.

We now prove that if \( R \) is noetherian and \( Q \) is irreducible then \( Q \) is primary. Suppose that \( ab \in Q \) but \( b \notin Q \). We wish to show that some \( a^n \in Q \). Now \( b \in [Q : a] \), so it is natural to consider the chain

\[
[Q : a] \subset [Q : a^2] \subset [Q : a^3] \subset \cdots
\]

Since \( R \) is noetherian this chain terminates. Say \([Q : a^n] = [Q : a^{n+1}] \). We claim that for this \( n \), \( (a^nR + Q) \cap (bR + Q) = Q \). Since \( Q \) is irreducible and \( b \notin Q \) this yields \( a^n \in Q \). Now suppose \( y \) is in this intersection; write \( y = a^nz + q \) with \( q \in Q \). Thus, \( ay \in Q \), whence \( a^{n+1}z \in Q \). Since \([Q : a^{n+1}] = [Q : a^n] \) we find that \( a^{n+1}z \in Q \), whence \( y \in Q \), as claimed.
Thus we have shown that every ideal \( I \) is the intersection of a collection of primary ideals \( Q_1, \ldots, Q_n \), and we can assume moreover that this collection is minimal (or irredundant), in the following sense:

- For all \( i, Q_i \not\supset \bigcap_{j \neq i} Q_j \).
- If \( i \neq j \) then \( \text{rad} Q_i \neq \text{rad} Q_j \).

The primes \( \text{rad} Q_i \) are called the associated primes for \( I \). Altho in general the \( Q_i \) are not uniquely determined, the associated primes are. We prove this using colon ideals.

In fact, the associated primes can be characterized as those colon ideals \([I : a]\) which happen to be prime.

We first establish several useful facts.

1. \([I : J] = R\) if and only if \( J \subset I \). Indeed, if \([I : J] = R\) then \( J = JR = J[I : J] \subset I \).

2. If \( P \) is prime and \( J \not\subset P \) then \([P : J] = P \). Indeed, \( J[P : J] \subset P \).

3. If \( Q \) is \( P \)-primary and \( J \not\subset Q \) then \([Q : J] \) is \( P \)-primary. Indeed, suppose that \( xyJ \subset Q \) but \( xJ \not\subset Q \). Since \( Q \) is \( P \)-primary we conclude that \( y \in P \subset \text{rad}[Q : J] \). Note: if \( y^nJ \subset Q \) and \( J \not\subset Q \) then \( y^{mn} \in Q \), for some \( m \). That is if \( Q \) is primary then \( \text{rad} Q \) is either \( \text{rad}[Q : J] \) or \( R \).

Now let \( I = Q_1 \cap \cdots \cap Q_n \) be a minimal primary decomposition of \( I \), with associated primes \( P_i = \text{rad} Q_i \). If \( a \in R - I \) then

\[ [I : a] = \bigcap_i [Q_i : a], \quad \text{rad}[I : a] = \bigcap_i \text{rad}[Q_i : a], \]

and \( \text{rad}[Q_i : a] = R \) or \( P_i \), depending on whether or not \( a \in Q_i \). Since primes are radical and irreducible we conclude that if \([I : a]\) is prime then \([I : a]\) is exactly one of the \( P_i \). Conversely, if we choose \( a \in \bigcap_{j \neq i} Q_j - Q_i \) then \( \text{rad}[I : a] = P_i \), whence \([I : a] = P_i \).

After we learn a bit more about localization we will also show that the primary components whose associated primes are isolated — that is, not containing any other associated primes — are uniquely determined.

We finish by observing the following corollary: if \( P_1, \ldots, P_n \) are the associated primes of 0 then \( \bigcup_i P_i \) is the set of zero-divisors of \( R \). Indeed, the set of zero divisors equals

\[ \bigcup_{a \neq 0} [0 : a] = \bigcup_{a \neq 0} \text{rad}[0 : a] = \bigcup_i P_i. \]