Answers to exam 1 — Math 4/5/7380 — Spring 05

Chapter 1

1. In how many ways can you seat 12 people at 2 round tables with 6 places at each?

Assuming the two tables are distinct, there are \( \binom{12}{6} \) ways to choose who sits at the first, and by default also who sits at the second. At each of the tables we have 6! seating arrangements, except that cyclic shifts are probably regarded as the same. Hence the total number of seating arrangements, in this interpretation, is \( \binom{12}{6} \cdot (5!)^2 \).

2. How many bits does \( 10^{100} \) have if written in base 2?

The number \( 10^{100} \) requires exactly \( 1 + \lfloor \log_2(10^{100}) \rfloor \) bits to express in binary. This is roughly \( 100 \log_2(10) \), or roughly 333 bits. (In fact, that is the exact number.)

3. Let \( B \) be a subset of \( A \). Suppose that \( |A| = n \) and \( |B| = k \). How many subsets of \( A \) intersect \( B \) in exactly 1 element?

To choose such a subset of \( A \) first choose an element of \( B \) and then a subset of \( A \setminus B \). There are \( k \cdot 2^{n-k} \) such options.

Chapter 2

4. What is the following sum?

\[
0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + \cdots + n \cdot \binom{n}{n}
\]

Give both an inductive and a combinatorial proof.

If we think of this sum combinatorially, it expresses the number of ways of picking a (nonempty) subset of \( n \), and then picking one element from that subset. This is equivalent to picking one of the \( n \) elements, and joining it with a subset from the remaining \( n-1 \) elements. There are \( n \cdot 2^{n-1} \) ways to do this.

We can verify this identity another way: since

\[
n \cdot (1 + x)^{n-1} = \frac{d}{dx} (1 + x)^n = \frac{d}{dx} \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} k \cdot \binom{n}{k} x^{k-1},
\]

we find that

\[
n \cdot 2^{n-1} = \sum_{k=0}^{n} k \cdot \binom{n}{k}.
\]

5. We select 38 even positive integers, all less than 1000. Prove that there exist two of them whose difference is at most 26.

Divide the range \( \{2, \ldots, 1000\} \) into 36 pigeonholes: \( \{2, \ldots, 28\}, \{30, \ldots, 56\}, \{58, \ldots, 84\}, \ldots, \{982, \ldots, 1000\} \). There must be at least one of these pigeonholes which contains at least two of the 38 even numbers. By design, such a pair differ by at most 26.

Chapter 3

6. In how many ways can you distribute \( n \) pennies to \( k \) children if each child is supposed to get at least 5 pennies?

First distribute 4 pennies to each of the \( k \) children, then distribute the remaining \( n - 4k \) pennies so that each child gets at least 1. There are \( \binom{n-4k-1}{k-1} \) ways to do this.
7. Prove that
\[ 1 + 2 \cdot \binom{n}{1} + 4 \cdot \binom{n}{4} + \cdots + 2^n \cdot \binom{n}{n} = 3^n. \]

Give both an inductive and a combinatorial proof.

The left-hand side counts the number of ways of picking a subset from a set of size \( n \), and then picking a subset of the subset. We can also accomplish this by labeling each of the elements with one of 3 labels: “not in the subset”, or “in the subset but not the subset of the subset”, or “in the subset of the subset”. There are 3\( n \) such labelings.

8. Use Stirling’s Formula to approximate \( \binom{3k}{k} \).

Since \( n! \sim (n/e)^n \sqrt{2\pi n} \), we have that
\[ \binom{3k}{k} \sim \frac{(3k/e)^{3k} \sqrt{6\pi k}}{(k/e)^{2k} \sqrt{8\pi^2 k^2}} = \frac{\sqrt{3}}{2} \cdot (27/4)^k / \sqrt{\pi k}. \]

Chapter 4

9. Which is larger, \( 2^{100} \) or \( F_{100} \)?

Since
\[ F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right) < \frac{1}{\sqrt{5}}(2^n + 1), \]
we should expect that \( F_n < 2^n \), at least eventually. In fact, \( F_1 < 2^1, F_2 < 2^2 \), and inductively
\[ F_{n+1} = F_{n-1} + F_n < 2^{n-1} + 2^n < 2 \cdot 2^n = 2^{n+1}. \]

10. Is it true that if \( F_n \) is prime then \( n \) is prime?

No, since \( F_3 = 3 \). But there is still “largely” true, since if \( m \mid n \) then \( F_m \mid F_n \). Unless \( n \) is prime or a power of 2, \( F_n \) cannot be prime.

11. How many subsets does \( \{1, \ldots, n\} \) have that contain no 3 consecutive integers? Find a recurrence.

Suppose \( S \) is such a subset. If \( S \) does not contain \( n \), then \( S \) is a subset of \( \{1, \ldots, n-2\} \). If \( S \) contains \( n \) but not \( n-1 \) then \( S = T \cup \{n\} \) where \( T \) is a subset of \( \{1, \ldots, n-2\} \) that contains no 3 consecutive integers. Otherwise, \( S = T \cup \{n-1, n\} \) where \( T \) is a subset of \( \{1, \ldots, n-3\} \) that contains no 3 consecutive integers. Hence if we let \( A_n \) denote the number of such subsets then we have established the recurrence
\[ A_n = A_{n-1} + A_{n-2} + A_{n-3}. \]

Hence the sequence is 1, 2, 4, 7, 13, 24, 44, 81, 149, …

If we let
\[ A(x) = A_0 + A_1 x + A_2 x^2 + \cdots \]
then our recurrence tells us that
\[ 1 + x + x^2 + (x + x^2 + x^3)A(x) = A(x), \]
whence
\[ A(x) = \frac{1 + x + x^2}{1 - x - x^2 - x^3}. \]
Chapter 5

12. Let \( S = \{1, 2, \ldots, 100\} \). Suppose we select a subset \( X \) of \( S \), randomly and uniformly (so that every subset has the same probability of being selected). What is the probability that

a. \( X \) has an even number of elements?

From the expansion of \((1 - 1)^n\) using the Binomial Theorem we know that the number of even-sized subsets is the same as the number of odd-sized subsets. Hence the probability that \( X \) is even is exactly \( \frac{1}{2} \).

b. both 1 and 100 belong to \( X \)?

There are \( 2^{98} \) subsets containing \( \{1, 100\} \) and \( 2^{100} \) altogether, and hence the probability of this event is \( \frac{1}{4} \).

c. the largest element of \( X \) is 50?

There are \( 2^{49} \) such subsets, and so the probability in this case is \( 2^{-51} \).

d. \( X \) has at most 2 elements?

The probability in this case is the ratio of \( 1 + 100 + 5050 \) to \( 2^{100} \).

13. We flip a coin \( n \) times, where \( n \geq 1 \). For which values of \( n \) are the following pairs of events independent?

a. The first coin flip is heads; the number of heads is even.

b. The first flip is heads; the number of heads is more than the number of tails.

c. The number of heads is even; the number of heads is more than the number of tails.

Chapter 6

14. Prove that every prime larger than 3 gives a remainder of either 1 or 5 when divided by 6.

If \( p \) divided by 6 leaves a remainder of 0, 2, or 4, then \( p \) is even. If \( p \) divided by 6 leaves a remainder of 0 or 3, then \( p \) is a multiple of 3.

15. Let \( a > 1 \) and \( k, n > 0 \). Prove that \( a^k - 1 \mid a^n - 1 \) if and only if \( k \mid n \).

If \( n = qk \) then

\[
 a^n - 1 = (a^k - 1)(1 + a^k + a^{2k} + \cdots + a^{(q-1)k}).
\]

On the other hand, if \( a^k - 1 \mid a^n - 1 \) and \( n = qk + r \), with \( 0 \leq r < k \), then

\[
 a^n - 1 = (a^k - 1) \cdot a^r + a^r - 1,
\]

whence \( a^k - 1 \mid a^r - 1 \). But \( a^r - 1 < a^k - 1 \), and thus \( a^r - 1 = 0 \) — that is, \( r = 0 \).

16. Prove that if \( a > 3 \) then \( a, a+2, \) and \( a+4 \) cannot all be primes. Can they all be powers of primes?

Suppose \( a \) is prime. If \( a \) divided by 3 leaves a remainder of 1 then \( a+2 \) is a multiple of 3. If \( a \) divided by 3 leaves a remainder of 2 then \( a+4 \) is a multiple of 3.

17. Use Euclid’s Algorithm to find integers \( x \) and \( y \) such that \( 25x + 41y = 1 \).

18. We are given \( n + 1 \) integers from the set \( \{1, 2, \ldots, 2n\} \). Prove that there exist two of these \( n + 1 \) numbers, \( a \) and \( b \), say, such that either \( a \mid b \) or \( b \mid a \).

19. Show that a number with 30 digits cannot have more than 100 prime factors.

Anything with more than 100 prime factors is larger than \( 2^{100} \), which is a number with more than 30 digits.