The Euler-Heun method

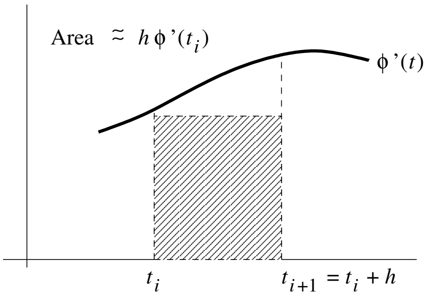
The Euler method is very simple, very intuitive, and produces an approximation that is as close to the exact solution as desired. However, the accuracy improves only *linearly* with the step size. In other words, it takes 10 times as many steps to achieve an approximation that is 10 times as accurate. By making a small adjustment we obtain a method whose accuracy improves *quadratically* — so that 10 times as many steps yields an approximation that is 100 times as accurate.

The idea here is to re-interpret Euler's method in terms of an integral equation for the solution. When we take this new point of view we will see that Euler's Method amounts to a Riemann sum approximation for the integral, and the improved Euler method — also called the Euler-Heun Method — amounts to the use of the Trapezoid Rule. Since the Trapezoid Rule error decreases quadratically with the step size, we gain a huge improvement.

So, let's spend some time understanding how to view a differential equation as an integration problem. If ϕ is a smooth function on an interval containing the points t_i and t_{i+1} then the Fundamental Theorem tells us that

$$\phi(t_{i+1}) = \phi(t_i) + \int_{t_i}^{t_{i+1}} \phi'(u)\, du. \tag{1}$$

So, we can approximate ϕ at t_{i+1} from information about ϕ at t_i, provided we can approximate the integral. For example, we can approximate ϕ' on the interval $[t_i, t_{i+1}]$ by the value at the left endpoint. This is of course the Euler Method.

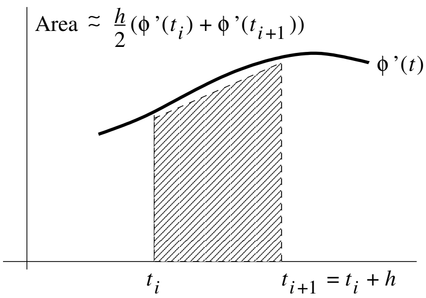


So, if ϕ satisfies the differential equation $y' = f(t, y)$ then we can obtain the approximation

$$\phi(t_i + 1) \approx \phi(t_i) + hf(t, \phi(t_i)).$$

Heun's Method uses the trapezoids instead of rectangles:

$$\phi(t_{i+1}) \approx \phi(t_i) + \frac{h}{2}(f(t, \phi(t_i)) + f(t, \phi(t_{i+1}))). \tag{2}$$



The only problem here is that the value we want to approximate, $\phi(t_{i+1})$, appears on the right-hand side. To get around this we use the Euler Method for approximating $\phi(t_{i+1})$, then use this value in the right-hand side of equation (2), to obtain an improved approximation:

$$\tilde{y}_{i+1} = y_i + hf(t_i, y_i) \tag{3}$$

$$y_{i+1} = y_i + \frac{h}{2}(f(t_i, y_i) + f(t_{i+1}, \tilde{y}_{i+1})). \tag{4}$$

Here is the table of values, with $h = 0.2$:

t_i	$\tilde{y}_i$	y_i
0.000000		1.000000
0.200000	1.200000	1.240000
0.400000	1.528000	1.576800
0.600000	1.972160	2.031696
0.800000	2.558035	2.630669
1.000000	3.316803	3.405416
1.200000	4.286500	4.394608
1.400000	5.513530	5.645422
1.600000	7.054506	7.215414
1.800000	8.978497	9.174806
2.000000	11.369767	11.609263

Here are the computed values at $t = 2$ for various step sizes.

h	$y(2)$	relative error
0.200000	11.609263	0.014336
0.040000	11.770465	0.000649
0.008000	11.777799	0.000027
0.001600	11.778100	0.000001

Notice that as we decrease h by a factor of 5 the relative error decreases by a factor of approximately 25. Compare these values with those obtained using the Euler Method. By doing twice as much work — two evaluations of f in each iteration — we roughly gain a lot of accuracy.

Further reading

Read section 8.2. Work thru problems 7–10, page 434.

Reading quiz

1. What is the integral form of a first-order differential equation $y' = f(t, y)$?
2. What integral approximation technique underlies the Euler-Heun Method?
3. What integral approximation technique underlies Euler's Method?
4. In what sense is the Euler-Heun Method an improvement over the Euler Method? Be precise!
5. How does changing the stepsize affect the error in Euler-Heun Method? Be precise!
6. How do you pronounce the names Freud, Euler, and Heun?
7. What is the difference between accuracy that improves linearly and accuracy that improves quadratically?

Exercises

Before dozing off, Goldilocks turned her attention to the Euler-Heun method. Her five approximations are tabulated below. Unfortunately, in all the confusion when the Bears came home, she forgot which column is which. Help her out: for each table, identify which column should be labeled t, y, $\tilde{y}$, $f(t, y)$, and $f(t, \tilde{y})$. *Justify your answers!*

1.06692	0.07	1.07146	0.90734	0.90924
1.16238	0.17	1.16551	0.97171	0.97245
1.26275	0.27	1.26409	0.99928	0.99933
1.36403	0.37	1.36340	0.98671	0.98681
1.46208	0.47	1.45951	0.93544	0.93635
1.55314	0.57	1.54889	0.85130	0.85352
1.63424	0.67	1.62871	0.74287	0.74656
1.70337	0.77	1.69702	0.61959	0.62456
1.75948	0.87	1.75275	0.49002	0.49588
1.80234	0.97	1.79559	0.36092	0.36721
1.83231	1.07	1.82580	0.23701	0.24333
1.85013	1.17	1.84402	0.12116	0.12722
1.85675	1.27	1.85113	0.01485	0.02046
1.85317	1.37	1.84808	−0.08149	−0.07641
1.84043	1.47	1.83585	−0.16804	−0.16352
1.81950	1.57	1.81541	−0.24538	−0.24141

−0.11074	4.22461	0.12	4.23074	−0.10461
−0.00522	4.21967	0.22	4.22522	0.00033
0.09027	4.22470	0.32	4.22973	0.09530
0.17670	4.23875	0.42	4.24330	0.18125
0.25491	4.26097	0.52	4.26509	0.25903
0.32570	4.29058	0.62	4.29430	0.32942
0.38975	4.32687	0.72	4.33025	0.39313
0.44773	4.36922	0.82	4.37227	0.45078
0.50019	4.41705	0.92	4.41981	0.50295
0.54767	4.46983	1.02	4.47233	0.55017
0.59065	4.52709	1.12	4.52935	0.59291
0.62953	4.58842	1.22	4.59047	0.63158
0.66473	4.65342	1.32	4.65527	0.66658
0.69658	4.72174	1.42	4.72342	0.69826
0.72540	4.79308	1.52	4.79460	0.72692
0.75149	4.86714	1.62	4.86851	0.75286
0.77510	4.94366	1.72	4.94490	0.77634
0.79646	5.02241	1.82	5.02354	0.79759
0.81580	5.10318	1.92	5.10420	0.81682
0.83330	5.18578	2.02	5.18670	0.83422

1.43587	−0.82404	−0.79174	0.08	1.45202
1.36962	−0.58571	−0.55923	0.18	1.38286
1.32428	−0.39028	−0.36857	0.28	1.33514
1.29611	−0.23003	−0.21223	0.38	1.30502
1.28201	−0.09863	−0.08403	0.48	1.28931
1.27945	0.00913	0.02110	0.58	1.28544
1.28635	0.09748	0.10730	0.68	1.29126
1.30101	0.16994	0.17799	0.78	1.30503
1.32203	0.22935	0.23595	0.88	1.32533
1.34826	0.27807	0.28348	0.98	1.35097
1.37877	0.31801	0.32245	1.08	1.38099
1.41279	0.35077	0.35441	1.18	1.41461
1.44969	0.37763	0.38062	1.28	1.45118
1.48895	0.39966	0.40211	1.38	1.49017
1.53014	0.41772	0.41973	1.48	1.53114
1.57291	0.43253	0.43418	1.58	1.57373

−0.04	−0.94034	0.85431	−0.86862	0.89017
0.06	−0.41108	0.79614	−0.35227	0.82554
0.16	0.02292	0.78443	0.07114	0.80854
0.26	0.37879	0.81083	0.41833	0.83060
0.36	0.67061	0.86848	0.70303	0.88470
0.46	0.90990	0.95176	0.93649	0.96505
0.56	1.10612	1.05604	1.12792	1.06694
0.66	1.26702	1.17755	1.28489	1.18649
0.76	1.39895	1.31319	1.41361	1.32052
0.86	1.50714	1.46042	1.51916	1.46643
0.96	1.59586	1.61714	1.60571	1.62207
1.06	1.66860	1.78166	1.67668	1.78570
1.16	1.72825	1.95256	1.73488	1.95587
1.26	1.77717	2.12870	1.78260	2.13142
1.36	1.81728	2.30913	1.82173	2.31136
1.46	1.85017	2.49309	1.85382	2.49492

0.84141	0.23	−0.28706	0.79793	−0.06965
0.83988	0.33	0.12059	0.81271	0.25647
0.86893	0.43	0.37537	0.85194	0.46029
0.91708	0.53	0.53460	0.90646	0.58768
0.97717	0.63	0.63413	0.97054	0.66730
1.04473	0.73	0.69633	1.04059	0.71706
1.11696	0.83	0.73521	1.11437	0.74816
1.19210	0.93	0.75950	1.19048	0.76760
1.26906	1.03	0.77469	1.26805	0.77975
1.34716	1.13	0.78418	1.34653	0.78734
1.42598	1.23	0.79011	1.42558	0.79209
1.50524	1.33	0.79382	1.50499	0.79506
1.58477	1.43	0.79614	1.58462	0.79691
1.66448	1.53	0.79759	1.66439	0.79807