Truncation error

The Euler and Euler-Heun methods are examples of *single-step methods*. This means that to determine y_{n+1} you only need to know t_n, y_n, h, and of course the function f in the differential equation $y' = f(t, y)$. Specifically, our methods take the form

$$y_{n+1} = y_n + h \cdot A(t_n, y_n, h, f), \tag{1}$$

where A is some algorithm for approximating the average slope $(y_{n+1} - y_n)/h$. The three most important examples are

$$\begin{aligned}
\text{Euler's method: } A(t_n, y_n, h, f) &= f(t_n, y_n) \\
\text{Euler-Heun method: } A(t_n, y_n, h, f) &= \tfrac{1}{2}(A_1 + A_2), \\
\text{where } A_1 &= f(t_n, y_n) \\
A_2 &= f(t_n + h, y_n + h \cdot A_1) \\
\text{Runge-Kutta method: } A(t_n, y_n, h, f) &= \tfrac{1}{6}(A_1 + 2A_2 + 2A_3 + A_4), \\
\text{where } A_1 &= f(t_n, y_n) \\
A_2 &= f(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}h \cdot A_1) \\
A_3 &= f((t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}h \cdot A_2) \\
A_4 &= f(t_n + h, y_n + h \cdot A_3)
\end{aligned}$$

(In contrast to these methods, there are *multistep methods*, which use several of the previous values y_i to predict the next one y_{n+1}.)

The single-step methods are closely related to Taylor series expansions. To understand their errors we distinguish focus on the main component of that error — the *local truncation error*. This is the error introduced by the approximation method A at each step. In other words, even if y_n equals the exact value $\phi(t_n)$ the approximation method will still only be able to approximate the next value y_{n+1}. We will let e_n denote the absolute value of the local truncation error. So, if $\phi(t)$ is the exact solution then

$$e_n = |\phi(t_n) - y_{n-1} - h \cdot A(t_{n-1}, \phi(t_{n-1}), h, f)|. \tag{2}$$

The *global error* E_n is the absolute difference between the correct value $\phi(t_n)$ and the approximate value y_n:

$$E_n = |\phi(t_n) - y_n| = |\phi(t_n) - y_{n-1} - h \cdot A(t_{n-1}, y_{n-1}, h, f)|. \tag{3}$$

If we compare formulas (2) and (3) we see the the relationship between the local and global errors is controlled by how sensitive $A(t, y, h, f)$ is to changes in the y-value.

If $\partial A/\partial y$ varies continuously over some bounded region R then in this region there is a constant L such that

$$|A(t, y, h, f) - A(t, z, h, f)| \leq L \cdot |y - z|. \tag{4}$$

Under such a condition (called a "Lipschitz condition") we can prove an important theorem relating the local and global errors.

To motivate this theorem remember that for the Euler and Euler-Heun methods, the global errors are on the order of h and h^2, respectively. The global error for the Runge-Kutta method are on the order of h^4. Later we will use power series to show that the local truncation errors for these three methods are h^2, h^3, and h^5, respectively. This relationship holds in general:

> *Suppose that for some initial value (t_0, y_0) we have a single-step method A which produces approximations (t_n, y_n) in some rectangle $[a, b] \times [c, d]$. Suppose also that on this rectangle we have the Lipschitz condition (4), and also a bound on the local truncation error of the form*
>
> $$e_n \leq Kh^{m+1} \tag{5}$$
>
> *for some positive constants K and m. If $Lh < 1$ then*
>
> $$E_n \leq (b - a)Kh^m. \tag{6}$$

The proof is very easy:

$$\begin{aligned}
E_n &= |\phi(t_n) - y_{n-1} - h \cdot A(t_{n-1}, y_{n-1}, h, f)| \\
&\leq |\phi(t_n) - y_{n-1} - h \cdot A(t_{n-1}, \phi(t_{n-1}), h, f)| + Lh \cdot |\phi(t_{n-1}) - y_{n-1}| \\
&\leq e_n + E_{n-1}.
\end{aligned}$$

If we repeat this $n-1$ more times we find that

$$E_n \leq e_n + e_{n-1} + \cdots + e_1 \leq nKh^{m+1} = (b-a)Kh^m.$$

Further reading

Truncation errors and roundoff are discussed in section 8.1. The Runge-Kutta method is introduced in section 8.3. Look at the tables on pages 454 and 460. Compute the errors at $t = 1$ and $t = 2$, for both methods and all step sizes. Compute the decrease in error for the Euler, Euler-Heun, and Runge-Kutta methods. Good practice problems are 7–13, page 456, and 7–13, page 461.

Reading quiz

1. What is local truncation error?
2. What is global truncation error?
3. How are local and global truncation error related?
4. What are the local truncation errors for the Euler, Euler-Heun, and Runge-Kutta method?
5. What are the global truncation errors for the Euler, Euler-Heun, and Runge-Kutta method?
6. How does changing the step size affect the error in Runge-Kutta Method? Be precise!
7. What is the difference between accuracy that improves linearly, quadratically, and quartically?

Assignment 12: due Monday, 20 March

This problem concerns the initial value problem $y' = 2t + 1 - y$, $y(0) = 1$.

1. Compute the exact solution $\phi(t)$. If we change the initial value to $y(t_0) = y_0$, what will the new solution be?
2. Apply the Euler-Heun and Runge-Kutta methods, with $h = 0.1$ and $h = 0.05$, for $0 \leq t \leq 2$. Make tables of your data for t-values in steps of 0.2. Include six columns, similar to those on page 460.
3. Use the general solution to the differential equation (from part 1) to estimate the local truncation errors for the last step of each of your four approximations. For each of the two methods, how much does the truncation error decrease from the first to the second step size?
4. Make accurate plots of the global errors, for each of the four approximations, and each of the t-values in your table. Use a single set of axes for all four plots.